University of the Witwatersrand
Advanced Techniques in Physics : 2002
Examination : June 2002

Instructions
This is the take - home question. It is to be handed out on Friday 8:30 21^{th} June.
There will be 24 hours to answer question 6.
Time for Question 6 : One day (take home conditions) Total Marks (Q6) = 70

3. This question requires some programming.
   Please remember to hand in your diskette with your script.

   White dwarfs are cold stars where the nucleo-synthesis process is completed. They consist of heavy stable nuclei and their electrons. In a simple model, they can be considered centrally symmetric and non-rotating (neglect magnetic fields). Their structure is then determined by the hydrostatic equilibrium between gravitational pressure seeking to compress the stellar material and Pauli pressure which resists this.

   Depending on the mass of the star, the nucleo-synthesis process is assumed to have run to completion leaving the star dominantly composed of a single stable nucleus terminating a fusion cycle, such as ^{12}C, or ^{56}Fe. The electrons are modelled as a $T = 0$ free Fermi gas, and they dominate the Pauli pressure term.

   Under these conditions, we can find two coupled first order differential equations for the radial mass and density distribution of the white dwarf.

   As in electrostatics, the gravitational force per unit volume at a given radial distance from the centre of the star is dependent on the amount of matter enclosed by a sphere of that same radius. Therefore

$$\frac{dP(r)}{dr} = \frac{dF(r)}{dV} = -\frac{Gm(r)}{r^2}\rho(r) \tag{1}$$

   where

$$m(r) = 4\pi \int_0^r \rho(r')r'^2\, dr' \tag{2}$$

   is the mass of the star to radius r. In differential form, this equation is

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r). \tag{3}$$

   This is our first differential equation mentioned above. By the chain rule, we would have our second differential equation,

$$\begin{aligned} \frac{d\rho(r)}{dr} &= \left(\frac{dP(\rho)}{d\rho}\right)^{-1} \cdot \frac{dP(r)}{dr} \\ &= -\left(\frac{dP(\rho)}{d\rho}\right)^{-1} \cdot \frac{Gm(r)}{r^2}\rho(r) \end{aligned} \tag{4}$$

   if we could just find $P(\rho)$, the equation of state for the white dwarf.

   The initial condition $\rho(r = 0) = \rho_c$ will determine the final mass M and radius R of the star, $M = m(R)$, by integration of the two coupled first order differential equations for $m(r)$ and $\rho(r)$.

The equation of state, $P(\rho)$, is in fact the Pauli pressure term mentioned above. We begin with the energy density of the electrons in units where $\hbar = c = 1$

$$\frac{E}{V} = 2\int_0^{p_f} \frac{d^3p}{(2\pi)^3}(p^2 + m_e^2)^{1/2} \tag{5}$$

where p_f is the Fermi momentum

$$p_f = (3\pi^2 n)^{1/3} \tag{6}$$

and n is the electron density

$$n = Y_e \frac{\rho(r)}{m_n}. \tag{7}$$

In the expression for the electron density, Y_e is the number of electrons per nucleon and m_n is the nucleon mass. Upon integration of equation 5, we have

$$\frac{E}{V} = n_0 m_e x^3 \epsilon(x) \tag{8}$$

where

$$\epsilon(x) = \frac{3}{8x^3}\left[x(1+2x^2)(1+x^2)^{1/2} - \log[x + (1+x^2)^{1/2}]\right] \tag{9}$$

and

$$x = \frac{p_f}{m_e} = \left(\frac{n}{n_0}\right)^{1/3} = \left(\frac{\rho}{\rho_0}\right)^{1/3} \tag{10}$$

with

$$n_0 = \frac{m_e^3}{3\pi^2} = 5.89\times 10^{29}\mathrm{cm}^{-3} \qquad \text{and} \qquad \rho_0 = \frac{m_n n_0}{Y_e} = 9.79\times 10^5 Y^{-1}\mathrm{gm\,cm}^{-3}. \tag{11}$$

Thus x is the electron density in units of n_0, chosen so that $x = 1$ corresponds to the Fermi momentum p_f being equal to the electron mass m_e (see equations 7 and 6).

Noting that $n = N/V$, we see that

$$\frac{\partial x}{\partial V} = -\frac{x}{3V} \tag{12}$$

According to thermodynamics, the pressure is the change in energy with volume, so that

$$\begin{aligned} P &= -\frac{\partial E}{\partial V} \\ &= -\frac{\partial E}{\partial x}\frac{\partial x}{\partial V} \\ &= \tfrac{1}{3} n_0 m_e x^4 \frac{d\epsilon(x)}{dx} \end{aligned} \tag{13}$$

Because the electron density scales with the density, $x \equiv x(\rho)$, equation 13 is indeed the equation of state. We can now evaluate

$$\begin{aligned} \frac{dP(\rho)}{d\rho} &= \frac{dP(x)}{dx}\frac{dx(\rho)}{d\rho} \\ &= Y_e \cdot \frac{m_e}{m_n} \cdot \frac{x^2}{3(1+x^2)^{1/2}} \end{aligned} \tag{14}$$

The two coupled first order differential equations then become :

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r) \tag{15}$$

and

$$\frac{d\rho(r)}{dr} = -\frac{G}{Y_e(m_e/m_n)} \cdot \frac{m(r)}{r^2} \frac{\rho(r)}{\gamma(x(\rho))}$$

where

$$\gamma(x) = \frac{x^2}{3(1+x^2)^{1/2}}. \tag{16}$$

Converting to dimensionless variables $\bar{r}$, $\bar{\rho}$ and $\bar{m}$ for the stellar radius, density and mass,

$$\frac{d\bar{m}}{d\bar{r}} = \bar{r}^2 \bar{\rho} \tag{17}$$

and

$$\frac{d\bar{\rho}}{d\bar{r}} = -\frac{\bar{m}\bar{\rho}}{\gamma \bar{r}^2}$$

where

$$r = R_0 \bar{r}, \quad \rho = \rho_0 \bar{\rho}, \quad m = M_0 \bar{m}, \quad x = \bar{\rho}^{1/3} \tag{18}$$

and the scale factors have been chosen to achieve unit coefficients in equation 17,

$$\begin{aligned} R_0 &= \left[\frac{Y_e(m_e/m_n)}{4\pi G \rho_0}\right]^{1/2} = 7.72 \times 10^8 Y_e \text{cm}, \\ M_0 &= 4\pi R_0^3 \rho_0 = 5.67 \times 10^{33} Y_e^2 \text{gm} \\ \rho_0 &= \frac{M_p n_0}{Y_e} = 9.79 \times 10^5 Y_e^{-1} \text{gm cm}^{-3}. \end{aligned} \tag{19}$$

Note that solutions for different values of Y_e may all be scaled from $Y_e = 1$.

a) As a partial approximate analytic solution, find the leading terms in the $\epsilon(x)$ and $\gamma(x)$ expressions for the non-relativistic ($x \ll 1$) and the relativistic ($x \gg 1$) limits and discuss the plausibility of the results. Using the result for small $\bar{\rho}$ with $\bar{m}$ and $\bar{r}$ finite, show the white dwarf has a well defined surface and find the functional dependence of $\bar{\rho}$ near the surface. **(10 marks)**

b) Solve numerically for $\bar{\rho}(\bar{r})$ and $\bar{m}(\bar{r})$. Mention any steps you could take, of a numerical nature, that would generate confidence in the numerical result. Calculate the stellar total mass and radius for centre densities $\bar{\rho}_c = \bar{\rho}(0)$ in the range 10^{-1} to 10^6. Does the behaviour of the solution correspond to your approximate analytical solution ? (explain). **(35 marks)**

c) Deduce the composition of Sirius B, $R = 0.0074(6)R_0$, $M = 1.05(3)M_0$. **(5 marks)**

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d) Investigate the behaviour of the mass and radius of the white dwarf for large central densities. You will find the mass approaches a limit and the radius collapses. Deduce the Chandrasekhar limit for the mass and central density of the white dwarf star? The destiny of such a large star would be a neutron star. **(10 marks)**

e) Explain the previous result using a simple model where the density profile is constant $d\bar{\rho}/d\bar{r} = k$. Calculate the total energy $U + W$ of the star where

$$U = \int_0^R \left(\frac{E}{V}\right) 4\pi r^2 dr \tag{20}$$

arises from the repulsive Pauli pressure and

$$W = -\int_0^R \frac{Gm(r)}{r^2} \rho(r) 4\pi r^2 dr \tag{21}$$

arises from the attractive gravitational potential. Show that for a given total mass M and at large densities (extreme relativistic limit) then both terms have a $1/R$ dependence and predict the Chandrasekhar collapse for large enough M. **(10 marks)**

(Your script will take the form of a report that addresses the questions. It will be accompanied by a diskette containing a well documented tested source code file and a graphic file for display of the theory curves.)

Total **Question 6 : [70 marks]**