University of the Witwatersrand

Advanced Techniques in Physics : Part 2 - 2004

Project 2 :
Study of the Deuteron

The Deuteron is the simplest nucleus of all. It is for nuclear physics what the hydrogen atom is for atomic physics. The binding energy of the deuteron is

$$B = \left[m(^1\mathrm{H}) + m_n - m(^2\mathrm{H})\right] c^2 = 2.22463(4)\ \mathrm{MeV}. \tag{1}$$

As the typical binding energy per nucleon is 8 Mev, the deuteron is about 8 times more weakly bound than a typical nucleon.

We would like a quantum mechanical model of the deuteron. As usual, we think of the simplest potential first. We choose a finite spherical square well in three dimensions (figure 1),

$$V(\mathbf{r}) = V(r) = \begin{cases} 0 & r \leq R_0 \\ -V_0 & r > R_0 \end{cases} \tag{2}$$

(Here $-V_0$ is the well depth which is an approximation to the strength of the nucleon-nucleon potential and R_0 may be taken as a size parameter for the deuteron.) We consider the lowest energy state of the deuteron which has $l = 0$.

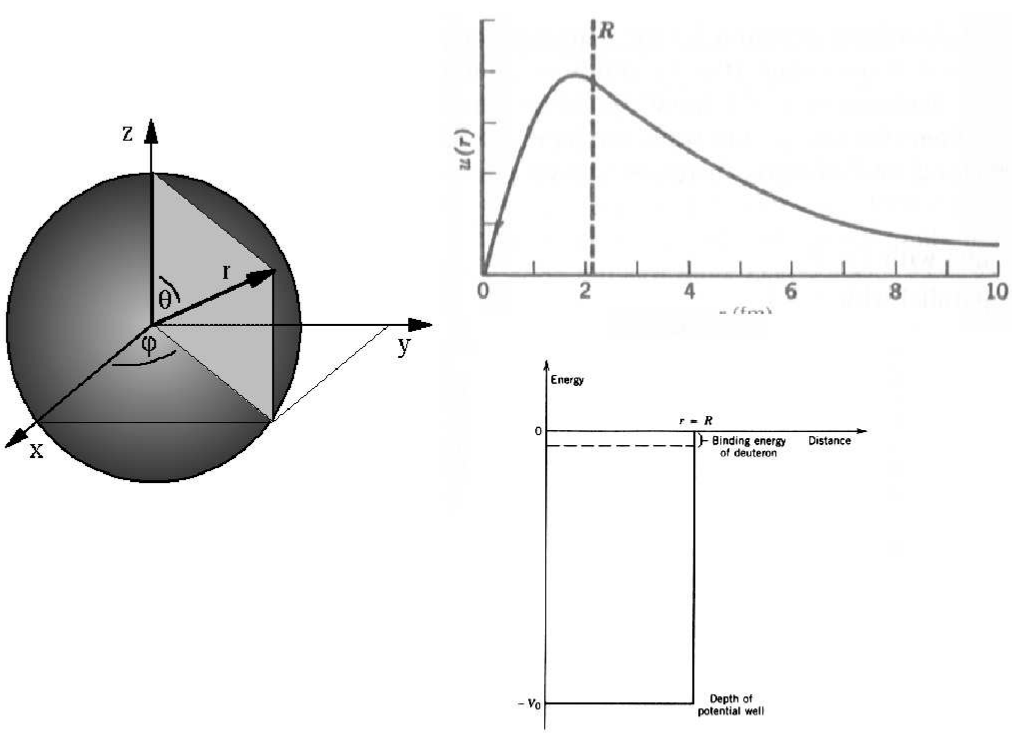

Figure 1: The finite spherical square well in three dimensions, as a model of the deuteron.

When we transform the Schrödinger Wave Equation from the cartesian co-ordinate system to the spherical polar co-ordinate system $(x, y, z) \longrightarrow (r, \theta, \phi)$, the radial part takes the form

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V(r)u(r) = Eu(r) \tag{3}$$

with $\psi(\mathbf{r}) = u(r)/r$.

We would like to investigate bound states where $E = -E_n < 0$ and $V_0 - E_n > 0$. The analytic expressions for the solutions are :

$$u(r) = \begin{cases} A\sin k_1 r & r \leq R_0 \\ Ce^{-k_2 r} & r > R_0 \end{cases} \tag{4}$$

with $k_1 = \sqrt{2m(V_0 - E_n)/\hbar^2}$ and $k_2 = \sqrt{2mE_n/\hbar^2}$. The continuity condition at $r = R_0$ leads to the transcendental equation

$$k_1 R_0 \cot k_1 R_0 = k_2 R_0 \tag{5}$$

Because of the structure of k_1 and k_2, this is effectively a family of relationships between the depth of the potential well V_0 and the spatial extent of the well R_0. From the the electron scattering experiments, we know that the size of the deuteron is $R_0 = 2.1$ fm. Inserting this value into the transcendental equation allows us to (numerically) deduce that the depth of the potential well is $V_0 = 35$ MeV.

It is clear therefore that the deuteron is only barely bound.

1. Assuming a well depth of $V_0 = 35$ MeV, solve the Schrödinger equation (eigenstates and wave functions) for the deuteron in the case of the square well, using the shooting method (to a fitting point). The method of shooting to a fitting point should be used.

   You should also as usual work in dimensionless variables,

   $$x = r/R_0, \quad \epsilon_n = E_n/V_0 \tag{6}$$

   with

   $$\gamma = \left(\frac{2mR_0^2 V_0}{\hbar^2}\right)^{1/2} \tag{7}$$

   and

   $$\frac{d^2u}{dx^2} = -\kappa^2 u = \begin{cases} -(1-\epsilon_n)\gamma u & x \leq 1 \\ \epsilon_n \gamma u & x > 1 \end{cases} . \tag{8}$$

   Don't forget that m is the reduced mass of the deuteron.

   The first order coupled ODE's will be

   $$\begin{aligned} u_1' &= u_2 \\ u_2' &= -\kappa^2 u_1 \\ u_3' &= 0 \end{aligned} \tag{9}$$

   Where the last one essentially handles the eigenvalue nature of the problem

   $$u_3 = \epsilon_n. \tag{10}$$

The boundary conditions will be

$$\begin{aligned} u_1(r=0) &= 0 \\ u_2(r=0) &= \delta \\ u_3(r=0) &= \text{unknown} \end{aligned} \tag{11}$$

and

$$\begin{aligned} u_1(r_{\text{max}}) &= 0 \\ u_2(r_{\text{max}}) &= \text{unknown} \\ u_3(r_{\text{max}}) &= \text{unknown} \end{aligned} \tag{12}$$

so that there is one unknown boundary condition at the origin and two unknown boundary conditions for large r. The homogeneous nature of the problem implies that δ is correct to a normalisation factor. The eigenvalue spectrum is determined sequentially by guessing start values of the shooting algorithm that begin at small numbers and gradually increasing subsequent guesses, thus probing the space of eigenvalues. Clearly the first guess at $u_3(r_{\text{max}})$ will be the same as $u_3(r=0)$.

(a) Verify that there is only one eigenvalue.

(b) Check wave functions of your numerical procedure against the analytic solutions. (Shape is sufficient, although it is possible to normalise both the analytic and the numerical solutions as usual.)

You should now have developed confidence in your numerical solution.

2. Now consider a more general potential. The phenomenological form

$$V_N(r) = \frac{-V_0}{1 + e^{(r-R)/a}} \tag{13}$$

is often used. It is simply a square well with rounded edges.

(a) Show that this solution evolves from that of the square well. This is another check of your programme.

(b) Is deuteron physics very sensitive to the shape of the potential ? Why do you think this is the case.

3. You may make any other observations regarding other possible tests of the numerical routine or explorations of the physics relevant to this model.