University of the Witwatersrand
Physics IIE (Engineering) : PHYS284 - 2006
Course Co-ordinator : Dr SH Connell

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Lectures : Mondays periods 3,4 and Fridays periods 1,23 in the Physics Building P114.

Tutorial period : Tuesdays period 5 in SH 312X

Laboratory :

1. Set Laboratories : Four praticals, Wednesdays periods 6-9.
   (Performed during weeks 2-9 of the course)
   (a) Statistics.
   (b) X-ray Structure determination.
   (c) e/m ratio of the electron.
   (d) The Hall effect.
2. Construct a Laser, Wednesdays periods 6-9.
   (Performed during weeks 10-14 of the course)

Notes : The notes provided below *must be supplemented* by your own notes taken during the lectures, study of the text books and proficiency with the tutorial problems.

Required Reading(*) :

A Beiser*	*Concepts of Modern Physics*	Chap's 1,5-7,9,10
R Serway and R Beichner	*Physics for scientists and engineers (with modern physics)*	as appropriate
L Solymar and D Walsh	*Lectures on the electrical properties of materials*	

Semester 1 : 25 double lectures

Lecturers		
Dr SH Connell	University of the Witwatersrand	Sections 1-5, 7
Dr A Forbes	National Laser Centre	Section 6

Course Introduction

This is a Modern Physics course for Electrical Engineers, taught by a physicist but targeted at engineers. The ethos of the course is to teach you how to think about nature in a totally new way. The field of electronics, which has implemented technologies for control, communication and information management, is growing so fast that it is expected to be fundamentally different by the time you graduate. Even the heavy current aspects are changing rapidly. This course hopes to equip you to be the electrical engineer of tomorrow, rather than the practitioner of today's technologies. To be an entrepreneurial innovator of tomorrow, you will have to understand the physics foundations of electrical engineering at a more abstract level. It is worthwhile reflecting that microelectronics has *absolutely no classical route* to its understanding. It was developed from pure theoretical considerations based on the very highest level of abstraction - quantum mechanics. Therefore, building intuition on the quantum (ghostly) nature of electrons is crucial.

The theoretical material in this course will arise out of modern experiments. These experiments first revealed that that nature appeared counter-intuitive when one went to extremes of relative velocity or physical dimension. Mathematical formulation of physics models allowed astoundingly powerful insights and extrapolations to be made from the ideas generated in these experiments. The course is therefore quite mathematical. It is hoped that you will enjoy the awesome vistas that this process will reveal, and that it will school your intuition in the physics phenomena which are the basis of the applications.

The final deliverable will be an understanding of micro- and nano-electronic devices at the level of their energy band structure, and how the energy band landscape is sculpted, both statically and dynamically, to achieve the myriad of devices that are deployed in communications and information systems today.

1. **Relativistic Mechanics** [8 lectures]
   (a) Relativity, reference frames
   (b) The Galilean Transformation
   (c) The failure of the Galilean Transformation
   (d) Special Relativity
   (e) The Lorentz Transformation
   (f) Time Dilation, The Doppler Effect
   (g) Length Contraction
   (h) The Twin Paradox
   (i) Electricity and Magnetism
   (j) The Relativity of Mass
   (k) Mass and Energy
   (l) Massles particles
   (m) General Relativity
   (n) Applications - GPS systems
2. **Introduction to Quantum Mechanics** [8 lectures]
   (a) Young's double slit experiment - Quantum mechanical behaviour
   (b) Wave Functions, Operators
   (c) Schrödinger's Time-Dependent Wave Equation
   (d) Calculating Observables
   (e) Schrödinger's Time-Independent Wave Equation
   (f) Simple Quantum Systems
      i. The particle in a box
      ii. The finite potential well
      iii. Barrier penetration, tunneling

    (g) Applications : The STM microscope, alpha decay, the quantum limit for the miniturisation of the classical computer

3. **Quantum Mechanics of Atoms** [8 lectures]

    (a) Introduction
    (b) A full Quantum Mechanical Model of the Atom
    (c) Solving the Schrödinger equation for hydrogen-like atoms,
    (d) Quantising intrinsic electron spin
    (e) Quantum numbers
    (f) Probability densities
    (g) Radiative transitions
    (h) Many-electron atoms
    (i) Symmetric / antisymmetric wave functions
    (j) Pauli's exclusion principle
    (k) Applications : Understanding the Periodic Table

4. **Statistical Mechanics** [5 lectures]

    (a) Introduction
    (b) Maxwell-Boltzmann Statistics
    (c) The Ideal Gas
    (d) Indistinguishability of particles and Quantum Statistics
    (e) Boson Statistics
    (f) Black-body radiation and Planck's Radiation Law
    (g) Fermion Statistics
    (h) Applications : Electrons in a metal - Ohm's Law, switches

5. **Modern materials** [1 lectures]

    (a) Nanomaterials
    (b) Superconductors

6. **Lasers** [5 lectures]

    (a) Introduction
    (b) Applications

7. **From Semiconductivity to Micro-electronics** [14 lectures]

    (a) Introduction, history, highlights, the future
    (b) Quantum Mechanical review
    (c) Crystal lattices, periodic potentials, surprising results
    (d) Band structure, mobility, effective mass, holes
    (e) Fermi statistics, charge carrier concentrations, dopants
    (f) Diffusion and drift of charge carriers

(g) Junctions, depletion regions, band bending, Fermi levels.

(h) Applications : Devices (diodes, transistors, solar cells, CCD's ...)

(i) Applications : Beyond Moore's law ... Quantum Computing and Communication

Credits

1. Material from the course textbook : A Beiser, *Concepts of Modern Physics* .
2. Material from *Open Questions in Relativistic Physics* (pp. 81-90), edited by Franco Selleri, published by Apeiron, Montreal (1998)

1 Relativistic Mechanics [8 lectures]

1.1 Relativity, reference frames

As an exercise in serendipity, imagine physics was actually poetry. Now, ... what would be the hallmark of a beautiful physics theory ?

Einstein was fascinated by this question. It was in fact the question that began to consume him totally towards the end of his life. Once again, now, for modern physicists, it is becoming a consuming question. Can we use this sense of beauty in a physics theory, as a way to develop the ultimate theory ?

The answer is very controversial, but many physicists now answer "Yes ! Indeed, we can !"

This journey we are about to make to discover special and general relativity is one of the early hints that symmetry is an important abstraction for identifying a sense of poetry in a physics theory.

The idea at first appears innocuous. You will be astounded later, as we go along, to find out how it totally overturned traditional thinking.

We want the laws of physics to be the same for all experimenters. It should not matter if you do your experiment on earth or the moon, or anywhere else. We will call the place where you do your observing from a "reference frame". All observers should be lead to the same laws of physics, even if their data appears different from different reference frames.

We will now proceed to formulate this insistence of ours in a mathematical way, so as to study for a variety of phenomena if we do indeed meet our "symmetry of reference frames" requirement.

Motion is relative. In fact the observation of motion (measurement) is a far more complex process than we may at first have imagined. For example, while walking down an aisle in a train your velocity relative to the station will be different depending on whether the train is moving through the station or halted at the station. However, in both cases you would be walking with the same velocity relative to the train. It is clear that the specification of the *Frame of Reference* with respect to which an observation was made is part of the description of the motion.

Are there different kinds of reference frames? Yes. A frame in which Newton's first law of motion holds identifies an inertial reference frame, when this law does not hold, we have a non-inertial reference frame.

Exercise 1.1
Think about how you would perform an experiment to detect whether you were in an inertial or a non-inertial reference frame.

How many inertial reference frame's are there? The answer is infinitely many. Any reference frame moving with a constant velocity with respect to an inertial reference frame, is itself an inertial reference frame

Exercise 1.2
Think about whether there could ever be an absolute reference frame. Think about whether any one inertial reference frame could be more "valid" than any other. Note - This question is not yet settled.

Galilean Relativity specifies the mathematical transformation between inertial reference frames in a way consistent with everyday experience.

1.2 The Galilean Transformation

Suppose there are two reference frames (systems) designated by S and S' such that the co-ordinate axes are parallel (as in figure 1). In S, we have the co-ordinates $\{x, y, z, t\}$ and in S' we have the co-ordinates $\{x', y', z', t'\}$. S' is moving with respect to S with velocity v (as measured in S) in the x direction. The clocks in both systems were synchronised at time $t = 0$ and they run at the same rate.

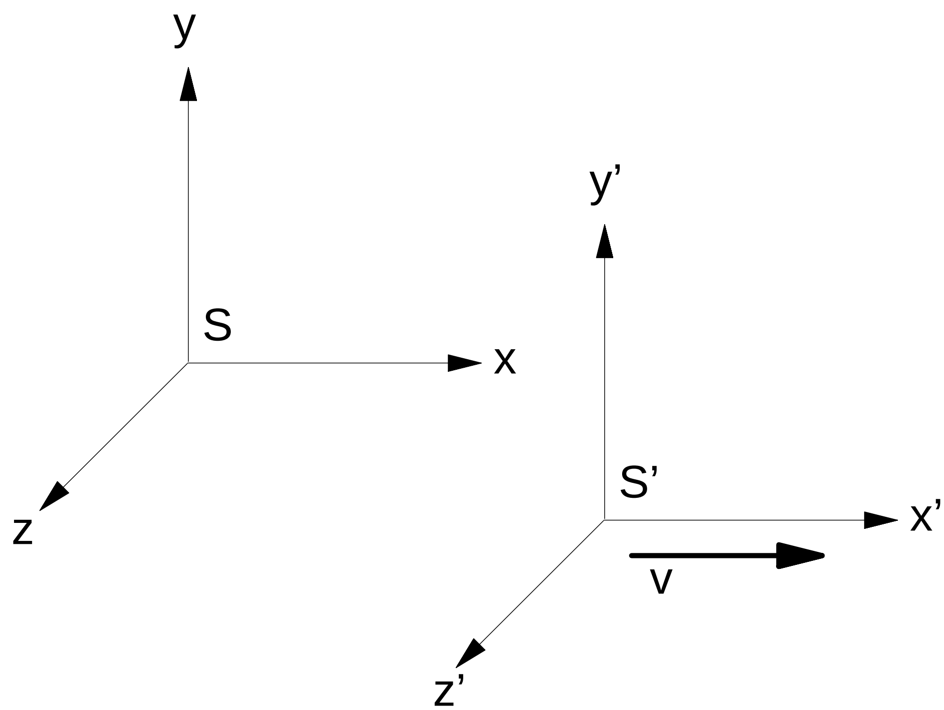

Figure 1: Reference frame S' moves with velocity v (in the x direction) relative to reference frame S.

We have the intuitive relationships

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \tag{1}$$

This set of equations is known as the Galilean Transformation. They enable us to relate a measurement in one inertial reference frame to another. For example, suppose we measure the velocity of a vehicle moving in the in x-direction in system S, and we want to know what would be the velocity of the vehicle in S'.

$$v'_x = \frac{dx'}{dt'} = \frac{d(x - vt)}{dt} = v_x - v \tag{2}$$

This is the result our intuition is familiar with.

We have stated the we would like the laws of physics to be the same in all inertial reference frames, as this is indeed our experience of nature. Physically, we should be able to perform the same experiments in different reference frames, and find always the same physical laws. Mathematically, these laws are expressed by equations. So, we should be able to "transform" our equations from one inertial reference frame to the other inertial reference frame, and always find the same answer.

Suppose we wanted to check that Newton's Second Law is the same in two different reference frames. (We know from experiment that this is the case.) We put one observer in the un-primed frame, and the other in the primed frame, moving with velocity v relative to the un-primed frame. Consider the vehicle of the previous case undergoing a constant acceleration in the x-direction,

$$\begin{aligned} f' = m'a' &= m'\frac{d^2x'}{dt'^2} \\ &= m'\frac{d}{dt'}\left(\frac{dx'}{dt'}\right) \\ &= m\frac{d}{dt}\left(\frac{d(x-vt)}{dt}\right) \\ &= m\frac{d(v_x - v)}{dt} \\ &= m\frac{dv_x}{dt} \\ &= ma = f \end{aligned} \tag{3}$$

Indeed, it does not matter which inertial frame we observe from, we recover the same Second Law of Motion each time. In the parlance of physics, we say the Second Law of Motion is invariant under the Galilean Transformation.

Exercise 1.3
In the tutorial, you show that the Law of Momentum Conservation holds regardless of the inertial frame a given collision is viewed in. This is done by specialising to a collision where all velocities are in the x-direction. How would you do this for a more general collision ?

So far so good !

We have Classical Mechanics, a beautiful theory, as it has an elegant independence of how you observe it. There is a sense of poetry in how ugly terms, arising from observation in a different frame, eventually drop away, until we are left with physical laws which are invariant under the Galilean Transformation.
But .... as time passes, it becomes clear we are in a fools paradise !
The first problem ...
Experiments on electric and magnetic fields, as well as induction of one type of field from changes in the other, lead to the collection of a set of equations, describing all these phenomena, known as Maxwell's Equations. You are already familiar with them. In vacuum they are

Maxwells Equations
in vacuo

$$\begin{aligned} \nabla.\mathbf{B} &= 0, \\ \nabla.\mathbf{E} &= 0, \\ \nabla\times\mathbf{B} &= \epsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}, \\ \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}. \end{aligned} \tag{4}$$

Now, these equations are considered to be rock solid, arising from and verified by many experiments. Amazingly, they imply the existence of a previously not guessed at phenomenon. This is the electromagnetic wave. Every electrical engineer, following Marconi, must appreciate this !

To see this in detail, take the time derivative of the second last equation and the curl of the last.

$$\begin{aligned} \frac{\partial}{\partial t}\nabla\times\mathbf{B} &= \epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}, \\ \nabla\times(\nabla\times\mathbf{E}) &= -\nabla\times\frac{\partial\mathbf{B}}{\partial t}. \end{aligned} \tag{5}$$

Now note that space and time derivatives commute

$$\frac{\partial}{\partial t}\nabla \times \mathbf{B} = \nabla \times \frac{\partial \mathbf{B}}{\partial t}, \tag{6}$$

so

$$\nabla \times (\nabla \times \mathbf{E}) = -\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{7}$$

Now, we use the identity

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \nabla . \mathbf{E} - \nabla^2 \mathbf{E}. \tag{8}$$

The second term of the above equation drops out due to the vanishing of the divergence of the electric field (the second of Maxwell's Equations). So, we finally have the three dimensional wave equation

$$\nabla^2 \mathbf{E} = \epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{9}$$

To see this is a wave equation, note the analogy in one dimension

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}. \tag{10}$$

which is solved by the wave function

$$y(x, t) = \sin(x - ct), \tag{11}$$

which represents a wave traveling along the x axis with velocity c.

It is clear therefore that Maxwell's Equations are highly predictive.

1. A diversity is unified in a simplicity. The various phenomena of radiowaves, microwaves, infrared, visible and ultra-violet light, X-rays and gamma rays are all electromagnetic waves, differing only in their frequency.
2. They all travel at the same speed.
3. Even that speed is specified : $c = 1/\sqrt{\epsilon_0 \mu_0} = 2.997 \times 10^8 \mathrm{m/s}$.
4. The speed appears independent of the source and the observer.

1.3 The failure of the Galilean Transformation

Maxwell's equations, which summarise electricity and magnetism, cause the Galilean Transformation to fail on two counts ....

1. They predict the speed of light is independent of the inertial reference frames instead of $(c' = c + v)$ as required by Galilean Relativity.
2. They are not invariant under the Galilean Transformation. (This is stated without proof in this course.)

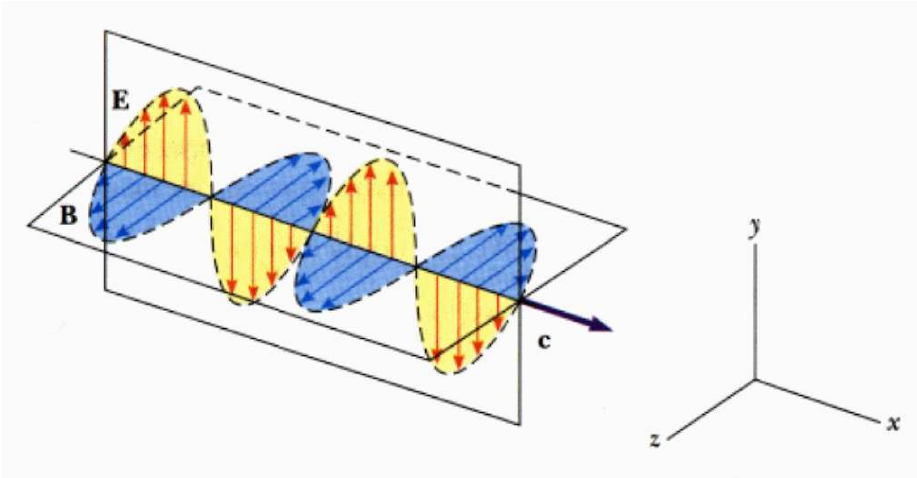

Figure 2: An electromagnetic wave traveling in vacuum, as required by equation 9

More sophisticated experiments (specifically, experiments on the behaviour of light and experiments that dealt with fast moving particles) indicated that Galilean Relativity was approximately correct only for velocities much smaller than the speed of light.

What a conundrum !

Shall we throw out all the theories of Electro-magnetsim ? This is hard to do. If these theories seem to have no flaw in their derivation, are firmly based in

experiments, their predictions are verified by further experiments, and we all use our cell phones with impunity, then its hard to fault Maxwell's equations !

The problem must lie somewhere else ....

But where ?

Enter **Special Relativity**, which was first developed by Einstein (1905). This theory treats inertial reference frames in a way

that is compatible with all measurements so far. Later on, came **General Relativity** which is able to deal with non-inertial reference frames and also to provide a geometrical way of dealing with gravity.

1.4 Experiments on the speed of light

It was thought in the 1800's that the frame-independence of the speed of light predicted by Maxwell's equations should be interpreted as the speed with reference to the *ether*, a mythical medium supporting the propagation of electro-magnetic waves. The ether would have to have the properties that material objects would pass through it with negligible friction. Taking the earth as a reference system, even if it were stationery with respect to the ether at some time, then six months later it would have a speed of $2v = 60$ km/s with respect to the ether. (Or correspondingly

a change in speed w.r.t. the ether of $2v = 60$ km/s w.r.t. its speed 6 months previously.) Many experiments were stimulated to measure the velocity of the earth through the ether. Figure 3 demonstrates schematically what one might expect for

measuring the speed of light c, which would be constant w.r.t. the ether, in an earth-bound laboratory moving with velocity v w.r.t. the ether. From earth, the ether would appear to be a "wind", so we may use the velocity addition formulae we are used to for an aircraft traveling in a wind.

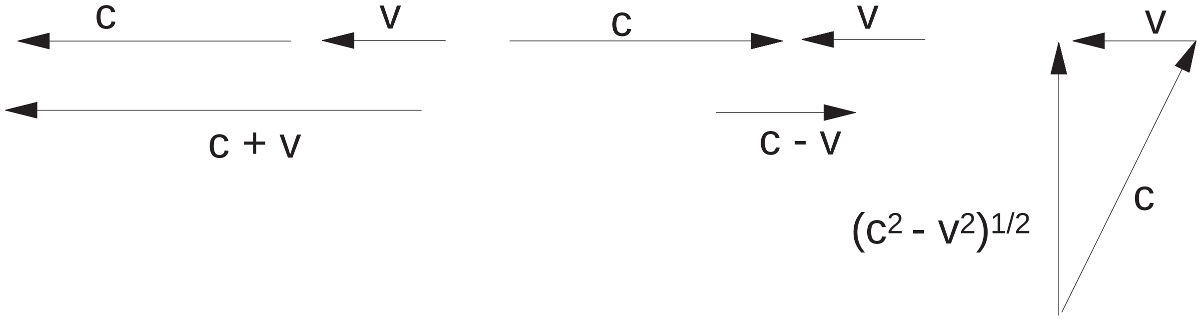

Figure 3: Down-wind, up-wind and cross-wind situations for the earth moving through the ether with velocity v.

It was not possible to detect differences in the speed of light accurately enough to do these experiments convincingly until late in the 1800's.

Exercise 1.4
What is the minimum accuracy for measuring the speed of light c in order to detect the motion of the earth through the ether, assuming the Galilean Transformation.

The most famous experiment designed to detect changes in the speed of light is now known as the Michelson-Morley experiment, performed in 1881. In this experiment, a Michelson Interferometer is used to produce an interference pattern from two beams which recombine at the detector after having been separated and sent on perpendicular paths by a half-silvered mirror. Assuming Galilean Relativity, interference fringes would pass the detector reflecting the changing optical path difference as the device was rotated through 90°. In this way the two perpendicular arms of the interferometer would experience the ether flowing past in different but correlated directions, leading to different optical path lengths in each arm.

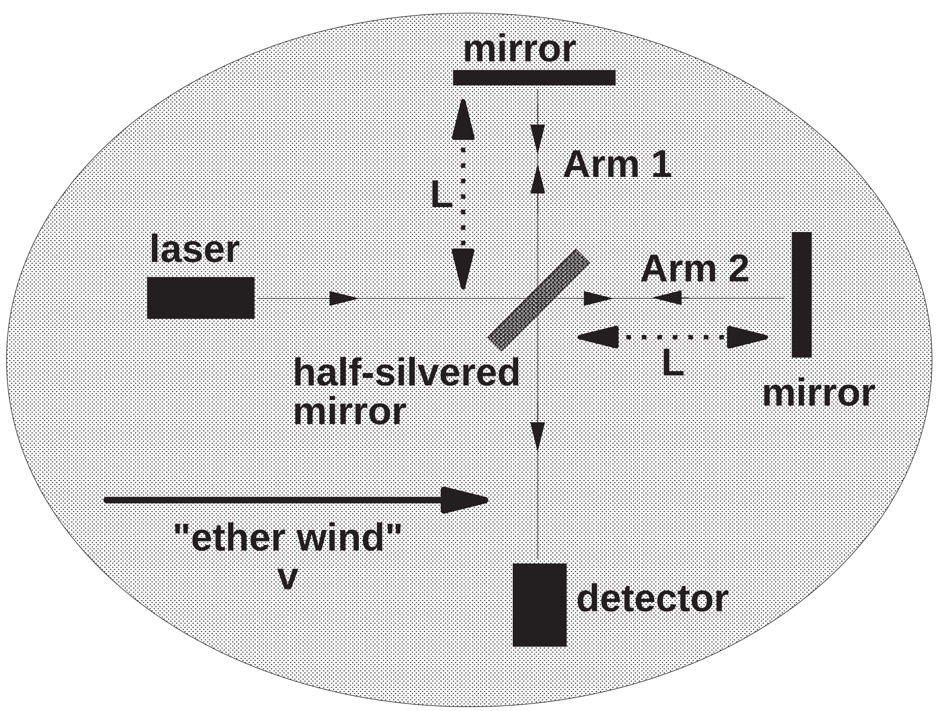

Figure 4: Schematic layout of the Michelson interferometer of the Michelson-Morley experiment.

To quantify this statement, we will calculate the time difference for the light beams to travel in Arm 1 and Arm 2 of the apparatus, once with Arm 2 parallel to the motion of the earth, and once with Arm 1 parallel to the motion of the earth.

Arm 2 parallel to the motion of the earth
The time difference between light traveling in Arm 1 and Arm 2 is

$$\begin{aligned} \Delta t = t_2 - t_1 &= \left[\left(\frac{l_2}{c+v} + \frac{l_2}{c+v} \right) - \frac{2l_1}{c^2 - v^2)^1/2} \right] \\ &= \frac{2l_2}{c(1 - v^2/c^2)} - \frac{2l_1}{c\sqrt{1 - v^2/c^2}} \end{aligned} \tag{12}$$

Arm 1 parallel to the motion of the earth
After rotating the apparatus by 90°, the time difference between light traveling in Arm 1 and Arm 2 is

$$\Delta t' = t_2' - t_1' = \left[\frac{2l_2}{c\sqrt{1 - v^2/c^2}} - \frac{2l_1}{c(1 - v^2/c^2)} \right] \tag{13}$$

The change in time difference between the un-rotated and rotated configurations is

$$\Delta t - \Delta t' = \frac{2}{c}(l_1 + l_2) \left(\frac{1}{(1 - v^2/c^2)} - \frac{1}{\sqrt{1 - v^2/c^2}} \right) \tag{14}$$

This can be simplified using the binomial expansion.
If $|x| < 1$, then $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$.
Since $c/v << 1$, we find

$$\frac{1}{(1 - v^2/c^2)} = (1 - v^2/c^2)^{-1} \approx 1 + \frac{v^2}{c^2} \tag{15}$$

and

$$\frac{1}{\sqrt{1 - v^2/c^2}} = (1 - v^2/c^2)^{-1/2} \approx 1 + \frac{1}{2}\frac{v^2}{c^2}. \tag{16}$$

Therefore

$$\begin{aligned} \Delta t - \Delta t' &\approx \frac{2}{c}(l_1 + l_2) \left(1 + \frac{v^2}{c^2} - 1 - \frac{1}{2}\frac{v^2}{c^2} \right) \\ &= (l_1 + l_2)\frac{v^2}{c^3}. \end{aligned} \tag{17}$$

Exercise 1.5
Check you can reproduce this result.

Using 500 nm light, an effective arm size in the Michelson interferometer of 11 m, the speed of light equal to $c = 2.997 \times 10^8$ m/s and the speed of the earth around the sun $v = 30$ km/s, we expect an optical path difference for the two arms of

$$\Delta d = 2.2 \times 10^{-7}\text{m}. \tag{18}$$

This would be an observable fringe shift of

$$\Delta \phi = \frac{\Delta d}{\lambda} \approx 0.44. \tag{19}$$

No such fringe shift was observed. It is now accepted that the ether concept, as well as the Galilean Transformation are wrong.

1.5 Special Relativity

In 1905, at the age of only 26, Einstein published his special theory of relativity. Regarding his theory, he wrote :

> The relativity theory arose from necessity, from serious deep contradictions in the old theory from which there seemed no escape.

Einstein based the Special Theory of Relativity on two postulates:

1. **The Principle of Relativity:** The laws of physics must be the same in all inertial reference frames.

2. **The constancy of the speed of light:** The speed of light in vacuum has the same value, $c = 2.997 \times 10^8$ m/s, in all inertial reference frames.

There is now no way to distinguish a preferred reference frame, and all reference frames are equivalent.

Both time and distance now have to be adjusted in such a way that the speed of light is $c = 2.997 \times 10^8$ m/s for all observers in inertial reference frames.

In your mind, embark on a radical concept. Time and distance determine how we think of space. Now, time and distance intervals are going to be "adjustable". It will actually be the properties of space-time that are changing. Space-time is not just an emptiness within which we erect co-ordinate systems. Space-time must be something fundamental, which already "knows" about "physics" and co-ordinate systems. Geometry, therefore, will assume a new role in physics.

The postulates of Special Relativity therefore force us to abandon completely our previous comfortable concepts of simultaneity, absolute length and absolute time. One can imagine synchronising two clocks at different positions in one reference frame by sending a light pulse across the distance d between them and then compensating for the light travel time $\Delta t = d/c$. However, to another observer, these distances and times will be different, so the simultaneity of events becomes dependent on the reference frame they are observed in.

But, so far the postulates are just a verbal wish list.

How do we express this mathematically ?

The guiding principle, in developing a new mathematical model, which will embody the new principles, is to start with the existing model (Galilean Relativity), and proceed from there in as simple a fashion as possible.

We therefore need to generalise the Galilean Transformation equations, in order to take into account that length and time intervals can shrink or grow depending on ones reference frame, so that the speed of light is always constant. Consider again the two reference frames (systems) designated by S and S' such that the co-ordinate axes are parallel (as in figure 5). In S, we have the co-ordinates $\{x, y, z, t\}$ and in S' we have the co-ordinates $\{x', y', z', t'\}$. S' is moving with respect to S with velocity v (as measured in S) in the x direction. The clocks in both systems were synchronised at time $t = 0$ when their origins overlapped, but of course, they no longer run at the same rate.

The cheapest mathematical increase in complexity is now

$$\begin{aligned} x' &= k(x - vt) \\ y' &= y \\ z' &= z \end{aligned} \tag{20}$$

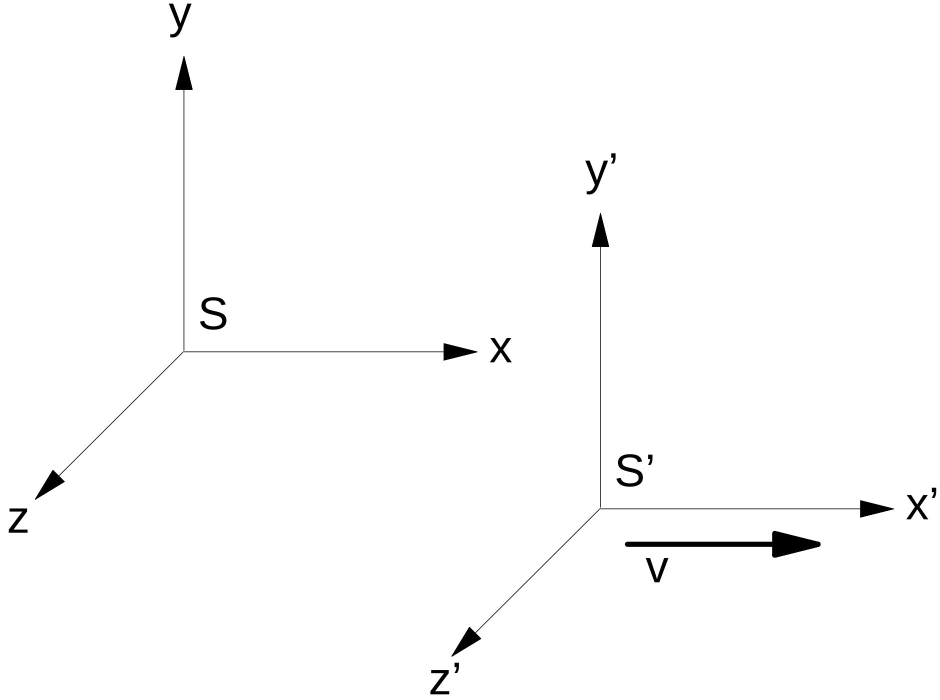

Figure 5: Reference frame S' moves with velocity v (in the x direction) relative to reference frame S.

which is linear in the co-ordinate variables and reduces to the Galilean Transformation when $k = 1$.

However, how should time be transformed ?

Suppose we invert the transformation, expressing co-ordinates of S in terms of S', by recognising that in this case the relative velocity of the two inertial reference frames now appears as $-v$.

$$\begin{aligned} x &= k(x' + vt') \\ &= k\left([k(x - vt)] + vt'\right) \\ &= k^2(x - vt) + kvt' \end{aligned}$$

so that

$$t' = kt + \left(\frac{1 - k^2}{kv}\right) x \tag{21}$$

Now, how do we evaluate k?

Simply by requiring that the speed of light, c, is the same in both reference frames. Therefore, in S

$$x = ct \tag{22}$$

and in S'

$$x' = ct' \tag{23}$$

Substitute for x' and t' in $x' = ct'$ to get

$$x = ct \left[\frac{1 + \frac{v}{c}}{1 - \left(\frac{1}{k^2} - 1\right)\frac{c}{v}}\right] \tag{24}$$

Clearly, the quantity in square brackets must be unity, so that

$$k = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{25}$$

Exercise 1.6
Fill in the missing steps, and make sure you can follow this derivation.

Now we can display the new transformation in all its glory :

$$
\begin{aligned}
x' &= \frac{(x - vt)}{\sqrt{1 - v^2/c^2}} \\
y' &= y \\
z' &= z \\
t' &= \frac{(t - \frac{vx}{c^2})}{\sqrt{1 - v^2/c^2}}
\end{aligned}
$$

1.6 The Lorentz Transformation

These co-ordinate transformations are known as the Lorentz Transformation. Note that the Lorentz Transformation reduces to the Galilean Transformation when $v \ll c$ so that $k = 1$ as required. Lorentz derived this transformation which made the Maxwell Equations of electromagnetism the same in all inertial reference frames in 1895. It was Einstein who is credited (in 1905) with discovering the full significance of these equations : The equations of the Lorentz Transformation define the relativity of space-time when measurements are made in different inertial reference frames.

The fact the the speed of light is "c" (constant) in all inertial reference frames implies a relativity in space-time. We expect length and time measurements to differ in different frames, in such a way that the speed of light is constant.

Because the relativistic effects become more obvious at velocities approaching the speed of light, a special symbol, "*beta*" defined as

$$\beta = v/c \tag{26}$$

is used in relativistic parlance. Clearly, for velocities such that $\beta \approx 0$, relativistic effects are negligible, and for velocities such that $\beta \approx c$, relativistic effects must be considered. The scale of relativity between reference fames is also an important characteristic. This is given the the special symbol "*gamma*" defined as

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{27}$$

Clearly, non-relativistic velocities $\beta \approx 0$ imply $\gamma \approx 1$, whereas extreme relativistic velocities imply $\gamma \gg 1$, as in figure 6

1.7 Length Contraction

Consider a rod lying along the x'-axis in system S' characterising an inertial reference frame. An observer in that same frame measures the rod, and notes its length is

$$L_0 = x'_2 - x'_1 \tag{28}$$

This is called the *proper length* of the rod as the measurement is made in the same frame as the rod itself. Now, what is the length L of rod for an observer in system S which characterises another inertial reference frame ?

$$L_0 = x'_2 - x'_1 \tag{29}$$

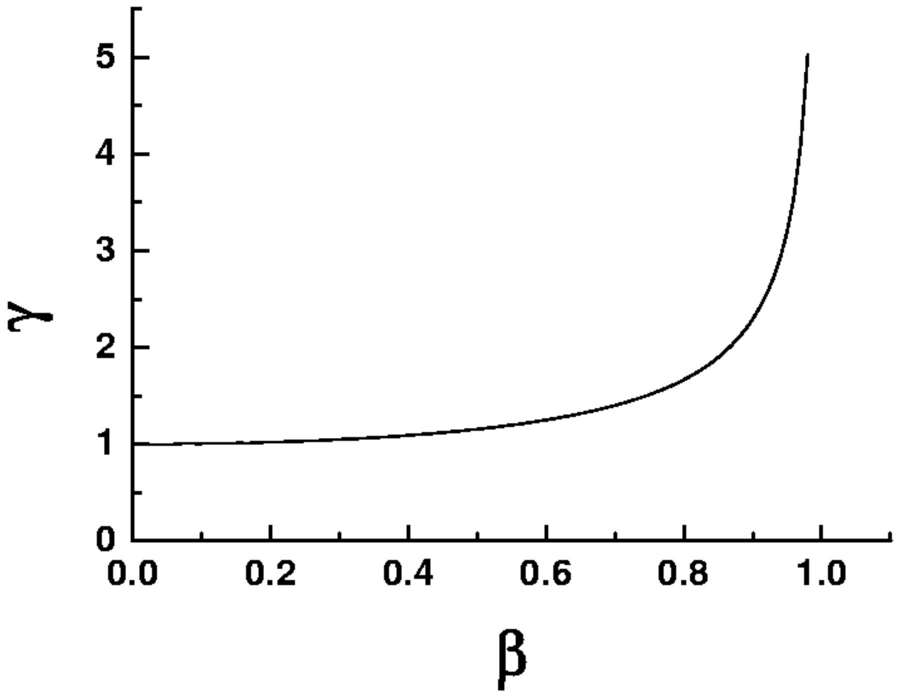

Figure 6: The gradual development of relativistic effects as $v \longrightarrow c$.

$$
\begin{aligned}
&= \left(\frac{(x_2 - vt_2)}{\sqrt{1-v^2/c^2}}\right) - \left(\frac{(x_1 - vt_1)}{\sqrt{1-v^2/c^2}}\right) \\
&= \left(\frac{(x_2 - vt)}{\sqrt{1-v^2/c^2}}\right) - \left(\frac{(x_1 - vt)}{\sqrt{1-v^2/c^2}}\right) \\
&= (x_2 - x_1)/\sqrt{1-v^2/c^2} \\
&= L/\sqrt{1-v^2/c^2}
\end{aligned}
$$

Note that we measure the rod in such a way that $t_2 = t_1$. So we find the length L of the rod, when viewed from a frame moving at a constant velocity with respect to the rod, appears contracted.

$$L = L_0\sqrt{1-v^2/c^2} = L_0/\gamma \tag{30}$$

The length L of the object in motion w.r.t. the observer always appears to be shorter than the length of the same object at rest w.r.t. the observer. This phenomenon is known as **length contraction**. The object only appears shorter in the direction parallel to its motion.

Exercise 1.7
Think about the reciprocity of this phenomenon. If two spacecraft passed each other, would observers in each craft see the same length contraction in the other craft ?

1.8 Time Dilation, The Doppler Effect

The calculation of an elapsed time interval must take into account that the moving clock will be in different positions w.r.t. the observer at the different time intervals. (Note the distinction of this case to that of length contraction.) It is therefore easier to use the *inverse* Lorentz Transformation obtained by replacing v with $-v$ and primed with unprimed quantities.

$$x = \frac{(x' + vt')}{\sqrt{1-v^2/c^2}} \tag{31}$$

$$
\begin{aligned}
y &= y' \\
z &= z' \\
t &= \frac{(t' + \frac{vx'}{c^2})}{\sqrt{1 - v^2/c^2}}
\end{aligned}
$$

Now the clock remains at the same point in the primed frame (moving w.r.t. the observer in the unprimed frame). The observer will see the time differences as

$$
\begin{aligned}
t &= t_2 - t_1 \\
&= \left(\frac{(t_2' + \frac{vx_2'}{c^2})}{\sqrt{1 - v^2/c^2}} \right) - \left(\frac{(t_1' + \frac{vx_1'}{c^2})}{\sqrt{1 - v^2/c^2}} \right) \\
&= \left(\frac{(t_2' + \frac{vx'}{c^2})}{\sqrt{1 - v^2/c^2}} \right) - \left(\frac{(t_1' + \frac{vx'}{c^2})}{\sqrt{1 - v^2/c^2}} \right) \\
&= (t_2' - t_1')/\sqrt{1 - v^2/c^2} \\
&= t_0/\sqrt{1 - v^2/c^2} \\
&= \gamma t_0
\end{aligned} \tag{32}
$$

where t_0 is the proper time as measured in the frame in which the clock is at rest. A clock that moves w.r.t. an observer ticks more slowly than a clock at rest w.r.t. an observer. Note that this means that all process (including those of life) seem to take place more slowly to an observer when they take place in a different inertial reference frame. This phenomenon is known as **time dilation**.

Example
A spacecraft is moving relative to the earth. An observer on earth finds that, according to her clock, 3601s elapse between 1pm and 2pm on the spacecraft's clock. What is the spacecraft's speed relative to the earth ?

Exercise 1.8
Consider the implications of the relativity of time by imagining a sequence of causally related events. Can Special Relativity imply a violation of causality?

The considerations of special relativity indicate we should re-investigate the phenomenon of the Doppler effect, which deals with the relative motion of a source and an observer. In the case of sound waves, we have

$$
\nu = \nu_0 \left(\frac{1 + v/c}{1 - V/c} \right) \tag{33}
$$

where

$$
\begin{aligned}
c &= \text{speed of sound} \\
v &= \text{speed of observer}
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
V &= \text{speed of source} \\
\nu &= \text{observed frequency} \\
\nu_0 &= \text{actual frequency}
\end{aligned}
$$

and the $\pm$ sign indicates decreasing or increasing the separation between the source and the observer. In the situation of sound waves, the Doppler Effect appears to violate the principle of relativity, as it matters whether the source or observer or both are moving.

Exercise 1.9
What is the resolution of this apparent paradox, and why would such a situation not occur for light sources ?

However, other new effects appear in the case of the relativistic Doppler Effect.

1. *Perpendicular relative motion :* The proper period of the light waves is $T_0 = 1/\nu_0$, in the reference frame of the source. In the reference frame of the observer, the period is $T = \gamma T_0$. So, the Doppler shifted frequency is
$$\nu_\perp = \nu_0/\gamma \tag{35}$$

2. *Receding relative motion :* In this case the observer travels a distance vt away from the source during the period of a wavelength. This means the wavefront will take a time vt/c longer to reach him. Accordingly, the new period of the wave is
$$T = t + \frac{vt}{c} = \gamma T_0(1 + v/c) \tag{36}$$
So, the Doppler shifted frequency is
$$\nu_\downarrow = \nu_0 \left(\frac{1 - v/c}{1 + v/c}\right)^{1/2} \tag{37}$$

3. *Closing relative motion :* Similar arguments lead to the result
$$\nu_\uparrow = \nu_0 \left(\frac{1 + v/c}{1 - v/c}\right)^{1/2} . \tag{38}$$

Exercise 1.10
Fill in any missing steps in the calculations above.

Example
A driver is caught going through a red light. The driver claims he actually saw a green ($\nu = 5.60 \times 10^{14}$Hz), not a red light ($\nu = 4.80 \times 10^{14}$Hz), as a result of the Doppler Effect. How fast would the driver have to have been driving ?

The Relativistic Doppler shift is an important tool in astronomy. The observed red shift of astronomical objects appears proportional to their distance from us, suggesting the entire universe is expanding. This proportionality is called Hubble's Law and is consistent with the Big Bang theory, whereby the Universe began from a quantum singularity about 15 billion years ago. Figure 7 shows two spectra for the binary star system, Mizar, taken two days apart. The simultaneous red and blue shifts of each star is clearly evident. The angular velocity of the system can be calculated.

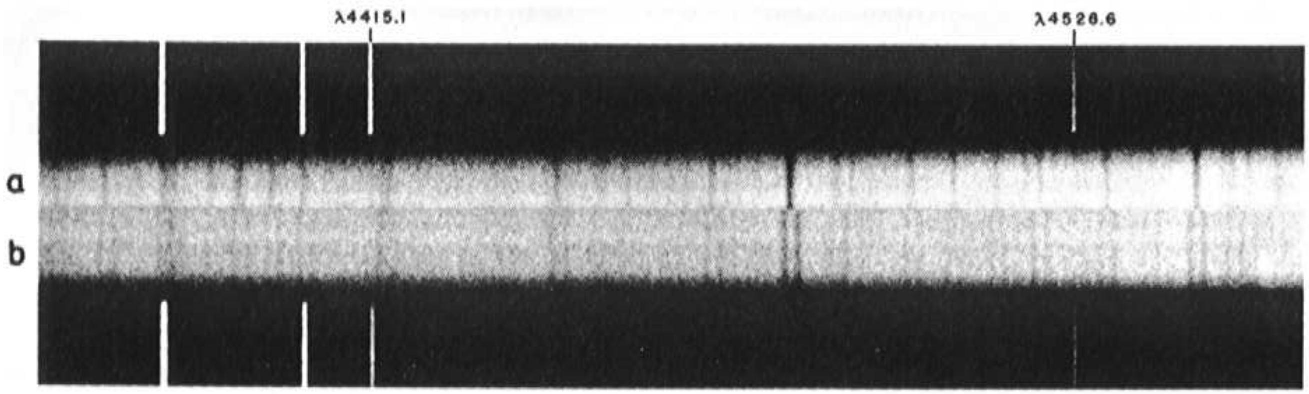

Figure 7: Two spectra for the binary star system, Mizar, taken two days apart.

One of the most dramatic confirmations of the phenomena of length contraction and time dilation is the profusion of muons reaching the surface of the earth. These muons are produced in cosmic ray collisions with the upper atmosphere at altitudes of 6 km and greater. A muon is a lepton, like an electron, but it has a larger mass ($m_\mu \approx 207 m_e$) and it is not stable ($\tau_\mu \approx 2.2\,\mu$s). The muons are produced in the upper atmosphere with speeds $v_\mu \approx 0.998c$.

Exercise 1.11
Show that muons could not be observed at the surface of the earth without considerations from Special Relativity.

1.9 The Twin Paradox

The twin paradox involves two identical clocks, one of which departs and the later returns, having traveled some distance in the intervening time at great speed. The clock which went on the voyage evidences less elapsed time. To make the problem more concrete, imagine two twins, Dick and Jane, each 20 years old. Dick departs on a space voyage, traveling at 0.8 c, to a star 20 light years away. When Dick returns, he is 50 years old, while Jane is 70 years old.

Exercise 1.12
Discuss this paradox.

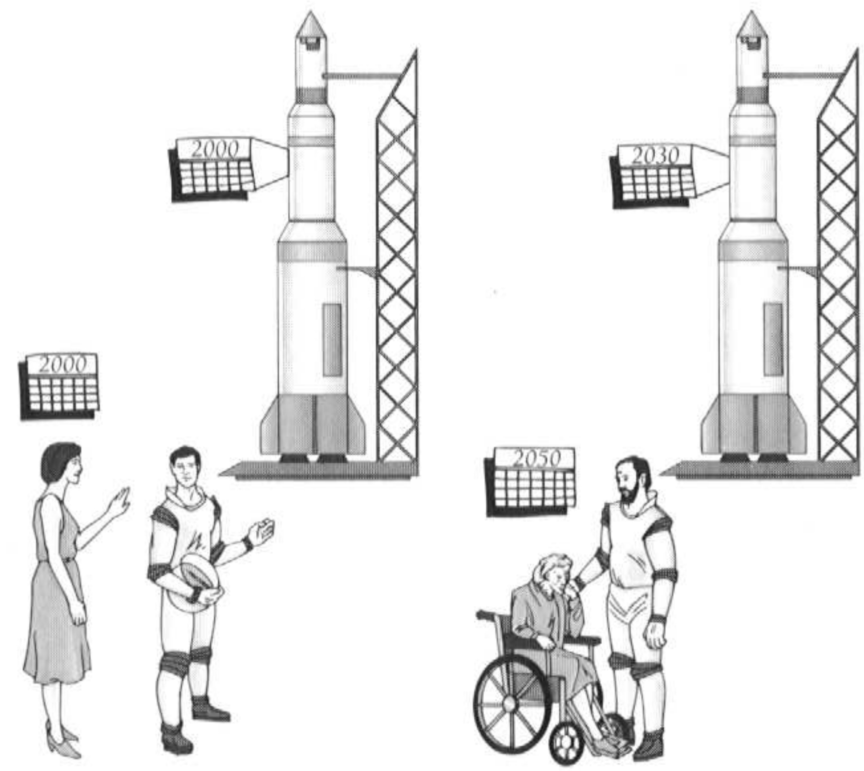

Figure 8: An astronaut who returns from space evidences the time dilation by his/her slower ageing.

1.10 Electricity and Magnetism

Special relativity connects the phenomenon of magnetism and electricity. Magentism arises from the motion of charge. Different observers will record different magnetic fields, if they are in different inertial frames. In some cases, the magnetism may disapera in a given inertial frame. However, the total electro-magnetic force will still be the same for all observers.

Electric charge is relativistically invariant.

That is, a charge Q remains the same regardless of the inertial reference frame it is observed in.

As an example, consider two parallel conductors, carrying current in the same direction. Normally we would associate a "magnetic" field with the moving charges in each of these conductors, and declare that the interaction of these magnetic fields led to the force of attraction between them, as in figure 9.

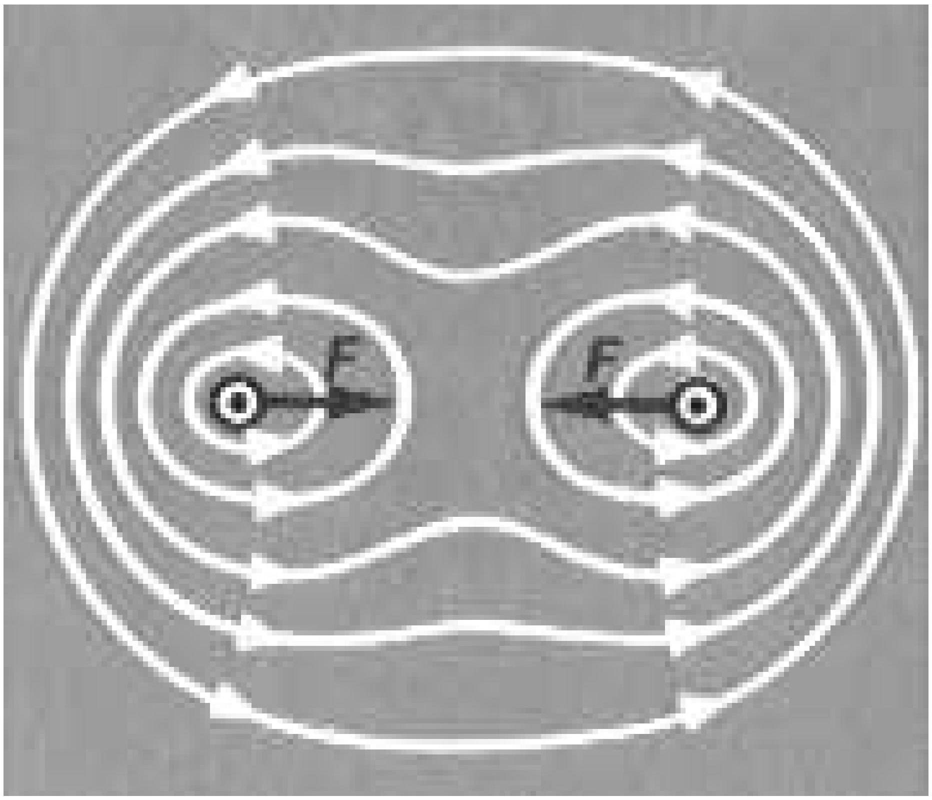

Figure 9: The resultant "magnetic field" around two current carrying conductors carrying current in the same direction.

The electric current in the conductors is manifested by the flow of electrons, against a background of stationery ions. The actual effective speed of an individual electron is only about 1 mm/s. However, there are about Avogadro's number of electrons flowing per cubic centimetre of conductor. The overall relativistic effect of is therefore quite large.

The discussion is simpler if one considers an imaginary conductor where both the positive ions and the negative electrons flow in opposite directions in each conductor. From the point of view of Special Relativity, the electrons and ions in conductors I and II and the laboratory are all characterised by a reference frame in which they are at rest.

Whenever the electrons or ions are viewed from a reference frame other than the one in which they are at rest, then the distances between those charges will be Lorentz contracted, resulting in an apparent increase in the number of charges per unit length, and therefore an excess of charge of that type, for that section of the conductor. In particular, in the reference frame of the electrons(ions) from conductor I, the ions(electrons) of conductor II will appear to be in excess. There will then be a Coulombic attraction between conductor I and II. The same argument will hold when viewing conductor I from conductor II. These ideas are illustrated in figure 10.

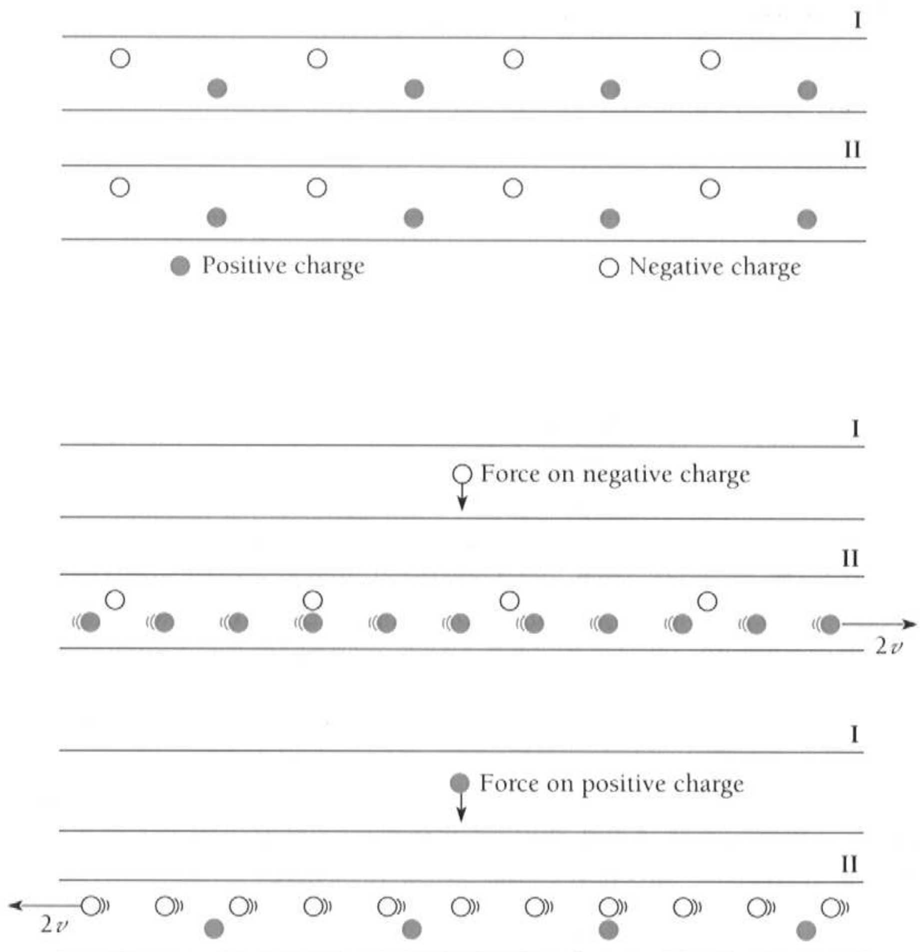

Figure 10: Two parallel current carrying conductors, viewed in three situations. Firstly, ignoring relativity. Secondly, viewing conductor II from a reference frame fixed on an electron in conductor I. Finally, viewing conductor II from a reference frame fixed on an ion in conductor I.

To tidy up the arguments, two further points must be mentioned. In the laboratory frame, the conductor appears electrically neutral, as from this frame both charge types are subject to the same Lorentz contraction of the distance between successive charges. Also, each circuit as a whole is electrically neutral when observed from any inertial reference frame, as flow in one section of the circuit will always be compensated by the reverse flow in the opposite section of the circuit.

1.11 The Relativity of Mass

We would like to insist that the Law of Conservation of Momentum is invariant under the Lorentz transformation. That is, during a collision, momentum will be conserved, regardless of the inertial reference frame from which we make our observation.

We will see that this will have the result that mass is relative to the inertial frame it is measured in.

To see this, imagine viewing a collision between two identical particles, A and B. We choose the reference frames in such a way that particle A has an initial velocity of $+v_y^A$ along the y axis in system S, while particle B has an initial velocity of $-v'^B_y$ along the y axis in system S' of equal magnitude. In otherwords,

$$|v_y^A| = |v'^B_y| \tag{39}$$

The particles A and B (and indeed the frames S and S') move relative to each other with velocity $-v$, as shown in figure 11. The collision is arranged so that momentum is only transferred in the y direction.

We will assume for the moment mass depends on velocity and follow through our study of this elastic collision in order to evaluate how to transform mass when moving between reference frames. Also, suppose we consider a particular collision where $v_y^A \ll v$ and $v_y'^B \ll v$. Then, from the point of view of reference frame S, the mass of particle A is $m^A(v_y^A)$ and that of particle B is $m^B(v)$. That is, the velocity dependence of the mass of particle B is dominated by the velocity of the reference frame S', from particle A's perspective.

After the collision, the particle A recoils with a velocity component $-v_y^A$ in the y direction as seen in the reference frame S, and particle B recoils with an equal but opposite velocity component $+v'^B_y$ as seen in the reference frame S'. This will be the case because of symmetry arguments.

However, if we view the motion of particle B from the reference frame S, we note that it will have a transformed velocity component in the y direction. We have already seen how velocities transform in Tutorial 1a :

$$\begin{aligned} v_x &= \frac{v'_x - v}{1 + \frac{vv'_x}{c^2}} \\ v_y &= \frac{v'_y\sqrt{1 - v^2/c^2}}{1 + \frac{vv'_x}{c^2}} \\ v_z &= \frac{v'_z\sqrt{1 - v^2/c^2}}{1 + \frac{vv'_x}{c^2}}. \end{aligned} \tag{40}$$

Therefore,

$$\pm v_y^B = \pm v'^B_y\sqrt{1 - v^2/c^2} = \mp v_y^A\sqrt{1 - v^2/c^2} \tag{41}$$

for before and after the collision respectively. Note that $v'^B_x = 0$.

Now, we have momentum conservation of the y-component of the momentum as viewed from system S.

$$m^A(v_y^A)v_y^A + m^B(v)v_y^B = -m^A(v_y^A)v_y^A - m^B(v)v_y^B \tag{42}$$

therefore, substituting for v_y^B we get

$$m^A(v_y^A)v_y^A - m^B(v)v_y^A\sqrt{1 - v^2/c^2} = -m^A(v_y^A)v_y^A + m^B(v)v_y^A\sqrt{1 - v^2/c^2}. \tag{43}$$

Solving for $m^B(v)$ we get

$$m^B(v) = \frac{m^A(v_y^A)}{\sqrt{1 - v^2/c^2}} \tag{44}$$

Now, as the particles are identical, we can drop the suffix on the mass. Finally, we allow $v_y^A \longrightarrow 0$, considering in the limit a glancing collision where the particle A in reference frame A is initially at rest. In this case $m^A(v_y^A) = m_0$. We can also use a notation that $m(v) = m$. The equation is now

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} = \gamma m_0 \tag{45}$$

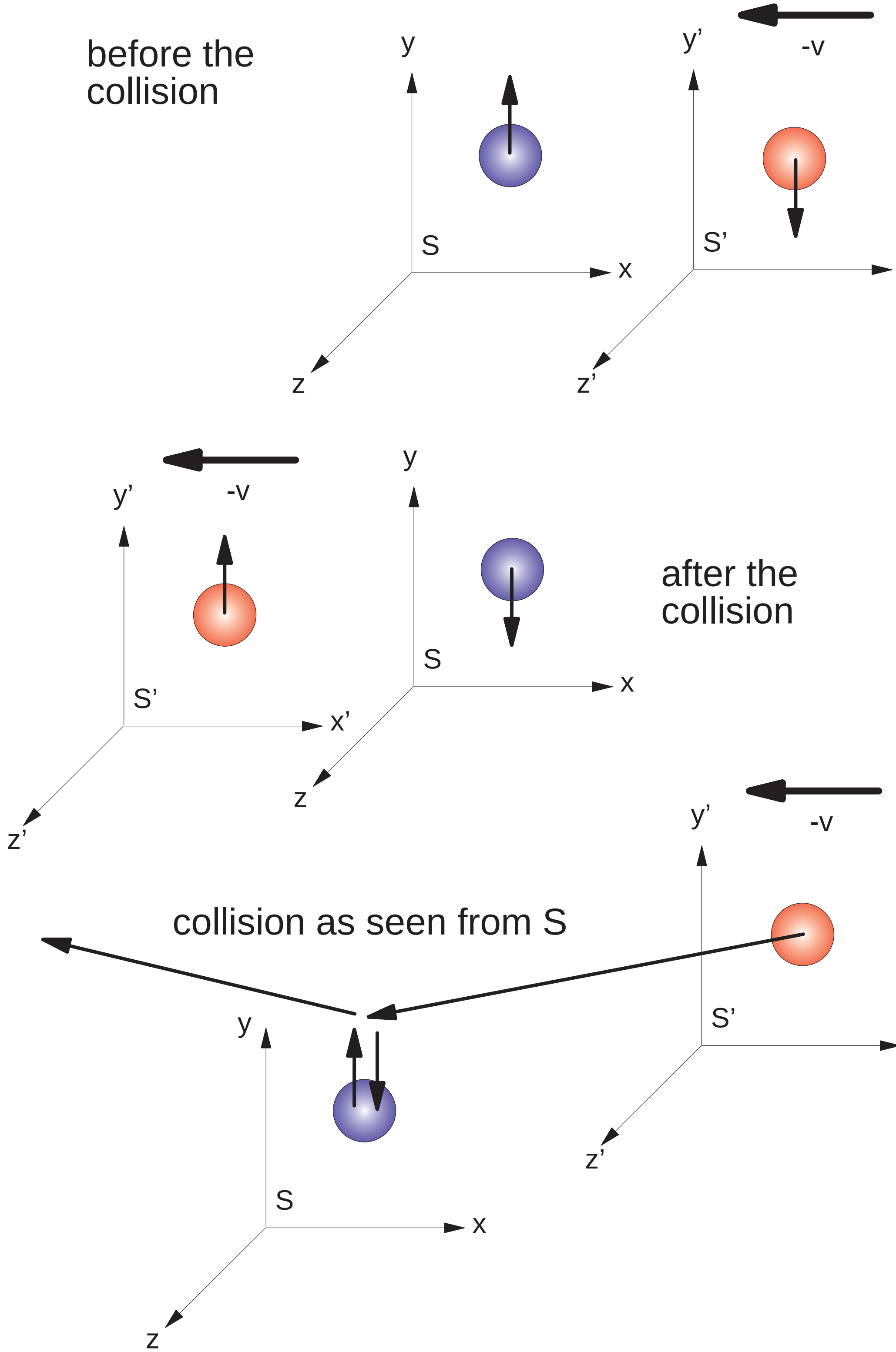

Figure 11: Reference frames for an elastic collision between two identical particles.

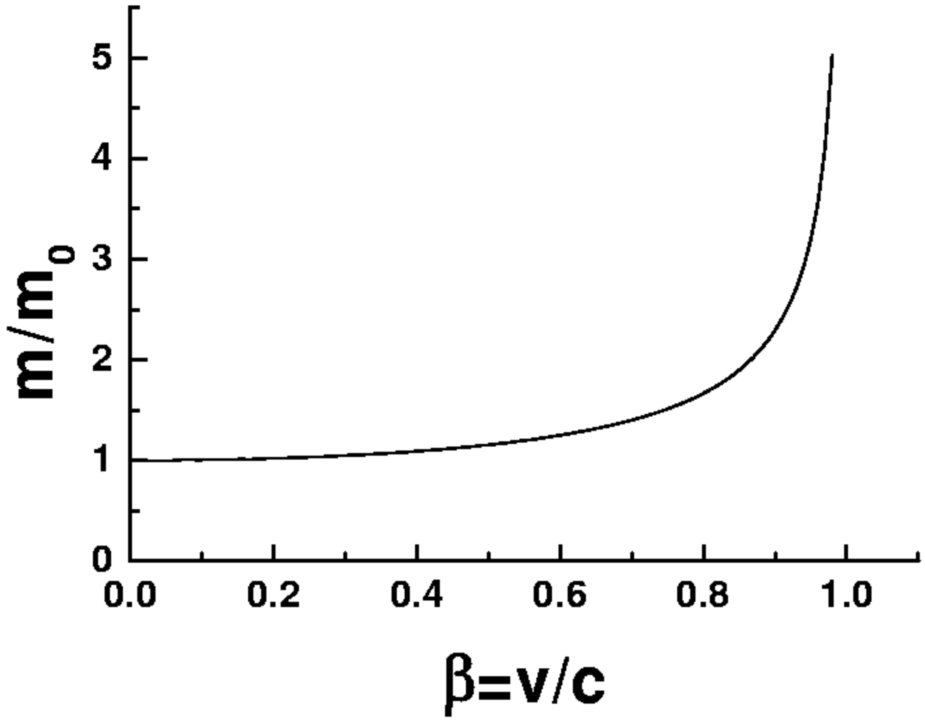

Figure 12: The relativistic increase of mass with velocity.

To continue to have conservation of momentum under conditions of special relativity, we find that mass will have to depend on velocity.

The relativistic mass increases only becomes significant for speeds $v \longrightarrow c$, as is depicted graphically in figure 12.

This theoretical prediction of relativistic mass increase with increasing velocity was first observed by Bucherer in 1908. Measurements of the e/m ratio for the electron showed this ratio diminished with increasing velocity.

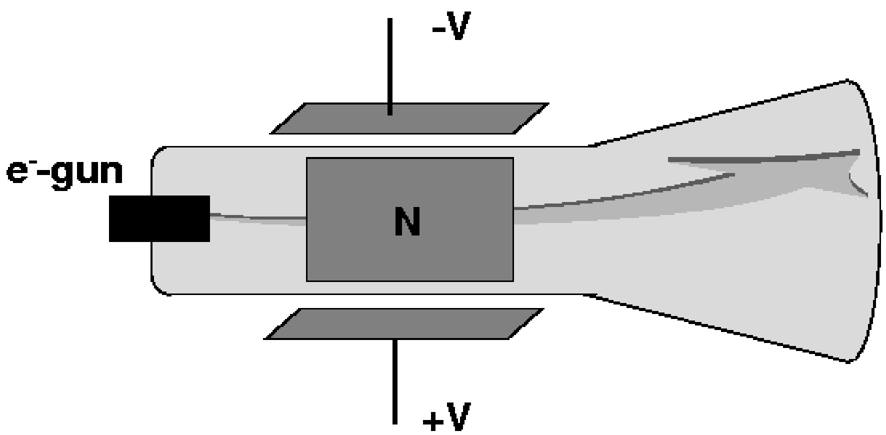

Figure 13: Apparatus consisting of crossed E and B fields for measuring the e/m ratio for the electron.

Example

Find the mass of an electron ($m_0 = 9.1 \times 10^{-31}$) kg whose velocity is $0.99c$.

The **Relativistic Momentum** can now be written down

$$p = mv = \gamma m_0 v = \frac{m_0 v}{\sqrt{1 - v^2/c^2}} \tag{46}$$

and the relativistic form of **Newton II** is

$$F = \frac{d}{dt}(mv) = \frac{d}{dt}(\gamma m_0 v) = \frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1 - v^2/c^2}}\right) \tag{47}$$

As both mass and velocity change with time

$$F = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt}. \tag{48}$$

1.12 Mass and Energy

Consider a electron, initially at rest. Work must be done on the electron (for example, by an electric field) in order to accelerate it and give it kinetic energy.

$$\begin{aligned} W &= E_k && (49) \\ &= \int_0^s F ds \\ &= \int_0^s \frac{d(mv)}{dt} ds \\ &= \int_0^v v\, d(\gamma m_0 v) \end{aligned}$$

Exercise 1.13
Fill in the missing steps, noting that γ is velocity dependent, and using integration by parts ($\int xdy = xy - \int ydx$), to get.

$$E_k = mc^2 - m_0 c^2 \tag{50}$$

The kinetic energy of a body is therefore equal to the increase of its mass (from the rest mass) multiplied by the square of the speed of light. Clearly, mc^2 and m_0c^2 represent energies with particular significance.

$$\textbf{Total Energy} = E = mc^2 \tag{51}$$

and

$$\textbf{Rest Energy} = E_0 = m_o c^2 \tag{52}$$

Therefore, the total energy may be written

$$\begin{aligned} E &= E_0 + E_k \\ &= mc^2 \\ &= \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} \end{aligned} \tag{53}$$

This equation implies an equivalence between mass and energy. Matter would seem to be a very compact for of energy. The law of Conservation of energy should be reworded to read the **The Law of Conservation of Mass-Energy**, in other words, both matter and energy are simultaneously conserved.

The relativistic expression for kinetic energy reduces to the normal classical expression at low speeds, $v \ll c$.

$$\begin{aligned} E_k &= mc^2 - m_0 c^2 \\ &= \gamma m_0 c^2 - m_0 c^2 \\ &= \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} - m_0 c^2 \\ &= m_0 c^2 \left(1 - v^2/c^2\right)^{-1/2} - m_0 c^2 \\ &\approx \frac{1}{2} m_0 v^2 \end{aligned} \tag{54}$$

Exercise 1.13
Apply the binomial approximation $(1+x)^n \approx (1+nx)$ for $X \ll 1$ to check the derivation above.

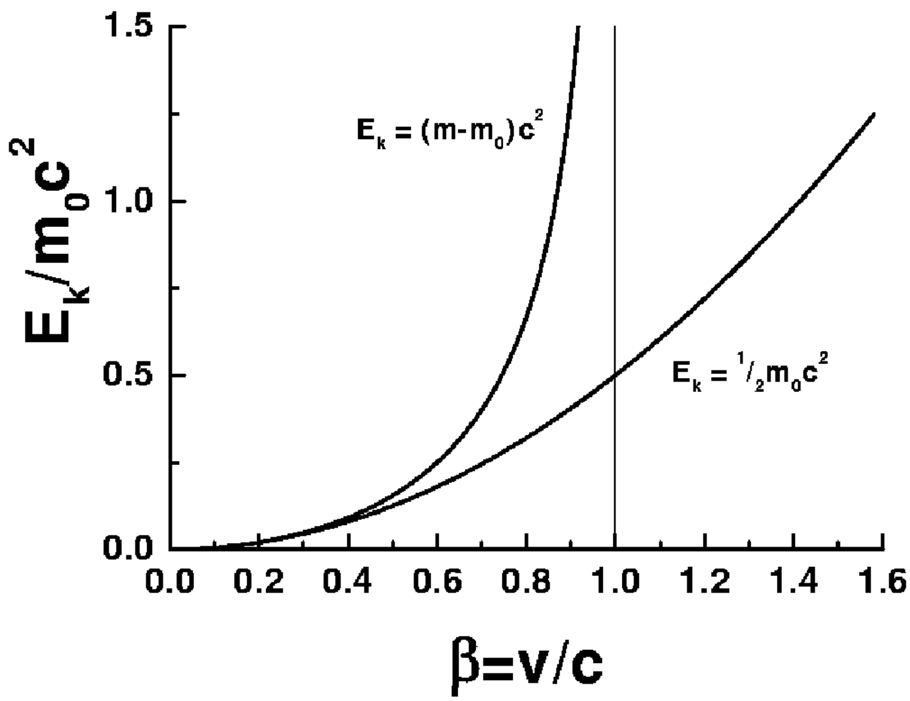

Figure 14: A comparison of relativistic and classical expressions for kinetic energy.

Figure 14 compares the relativistic and classical expressions for kinetic energy. Note a particle with rest mass would need infinite energy to travel at the speed of light. As work is done on a particle to increase its kinetic energy, the particle moves faster. As its velocity approaches the speed of light, more and more of the kinetic energy is manifested as an increase in mass of the particle, rather than an increase in its velocity.

The recovery of classical mechanics as a limiting case of relativistic mechanics is a general feature of Special Relativity. Generally, the theory of choice depends on the degree of accuracy required and the velocities involved.

Example
The solar constant (rate at which solar energy from the sun reaches the earth) is 1.4 kw/m^2 (normal area). Calculate the energy loss of the sun due to the solar radiation. (Hint : the mean radius of the earth's orbit is $R_E = 1.5 \times 10^{11}$m)

1.13 Massless particles

From the expression for the total energy

$$E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} \tag{55}$$

and the momentum

$$p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}} \tag{56}$$

of a relativistic particle, it is clear that

$$E = p = o \qquad \text{if} \qquad m_0 = 0 \qquad \text{and} \qquad c \neq 0 \tag{57}$$

except if the particle travels with the speed of light ($v = c$), as then

$$E = \frac{0}{0} \qquad \text{and} \qquad p = \frac{0}{0} \tag{58}$$

which are indeterminate (can have any value). Thus, the only particles that can have zero rest mass are those which travel at the speed of light. In addition we can rewrite the last two equations as the single equation

$$E^2 - p^2 c^2 = m_0^2 c^4 \tag{59}$$

Exercise 1.14
Show this.

Thus

$$\textbf{Massive particles} \qquad E^2 = m_0^2 c^4 + p^2 c^2 \tag{60}$$

and

$$\textbf{Massless particles} \qquad E = pc \tag{61}$$

The photon ($E_\gamma = hf$) is a particle with zero rest mass but which nonetheless carries a momentum $E_\gamma = pc$.

Exercise 1.15
Show the consistency of this with de Broglie's relation and ruminate on your introduction to quantum mechanics via the wave particle-duality in your 1st year of physics.

1.14 General Relativity

Special relativity is generalised to accommodate non-inertial reference frames. This is done via the principle of relativity, illustrated in figure 15 and stated below.

> It is not possible to distinguish (in a closed system) between the effects produced by a gravitational field and those produced by an acceleration of the closed system

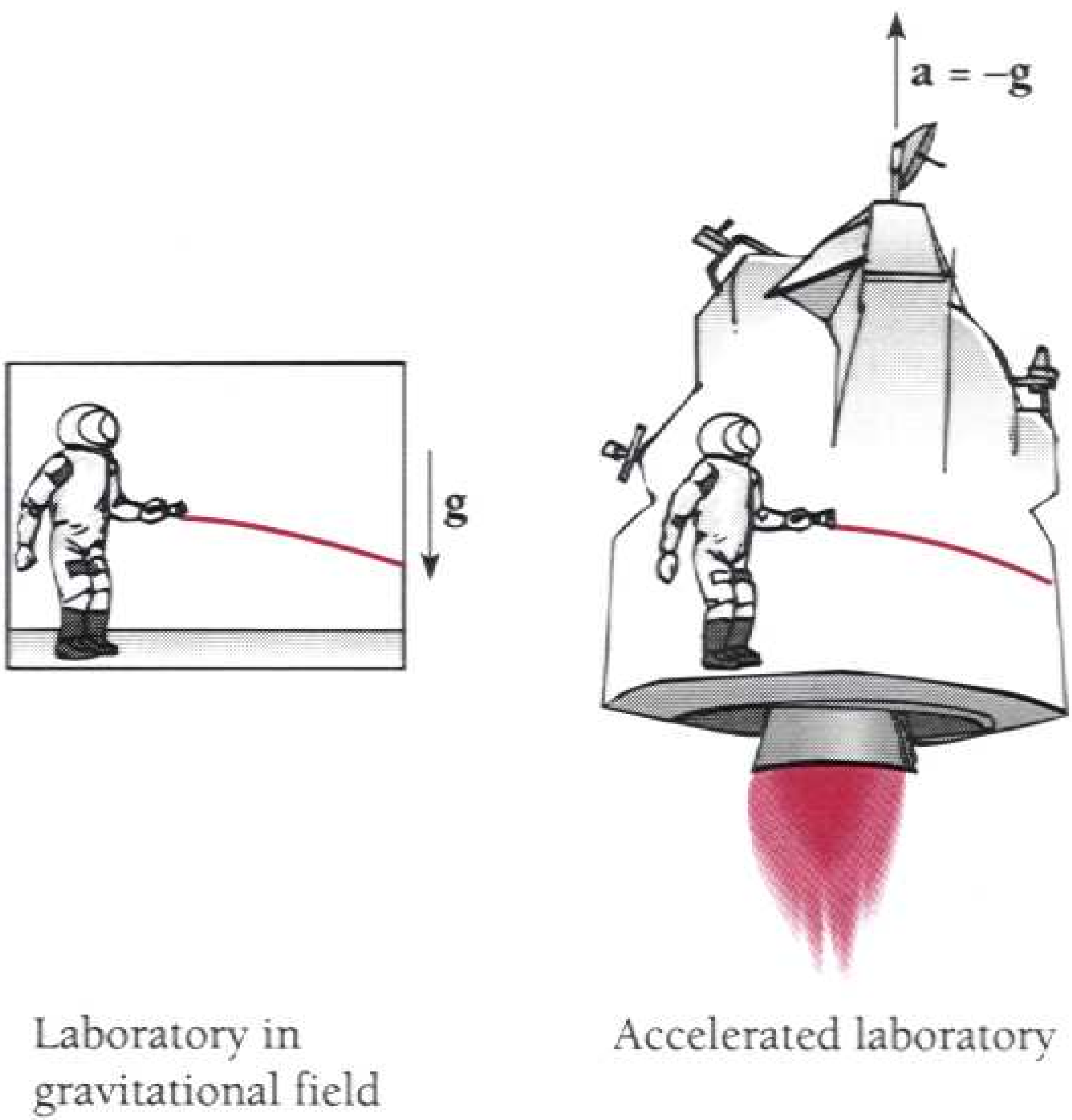

Figure 15: Graphic depiction of the principle of equivalence.

This principle allows one to replace the effects of gravity by equivalent effects based on the geometry of space-time. Once gravity is "abolished" in this way, and there is no "force of gravity" then all (gravitating) objects will have motions described by Newton's First Law. That is, those in motion will continue in a straight line at constant velocity. However, "straight line" now means only locally straight (locally parallel to a co-ordinate axis in space). However, the geometry of space is now "warped" (no longer Euclidean) in such a way that the objects actual trajectory is "similar" to that calculated in the classical way. Einstein wrote down a Field Equation which allowed the warping of the geometry of space-time to be calculated given a certain mass distribution.

The trajectory of the moon around the earth is locally straight in a space-time region warped by the presence of the earth's mass. Such "straight lines" are called geodesics, defined as the shortest distance between two points in a curved space. This is illustrated in figure 16.

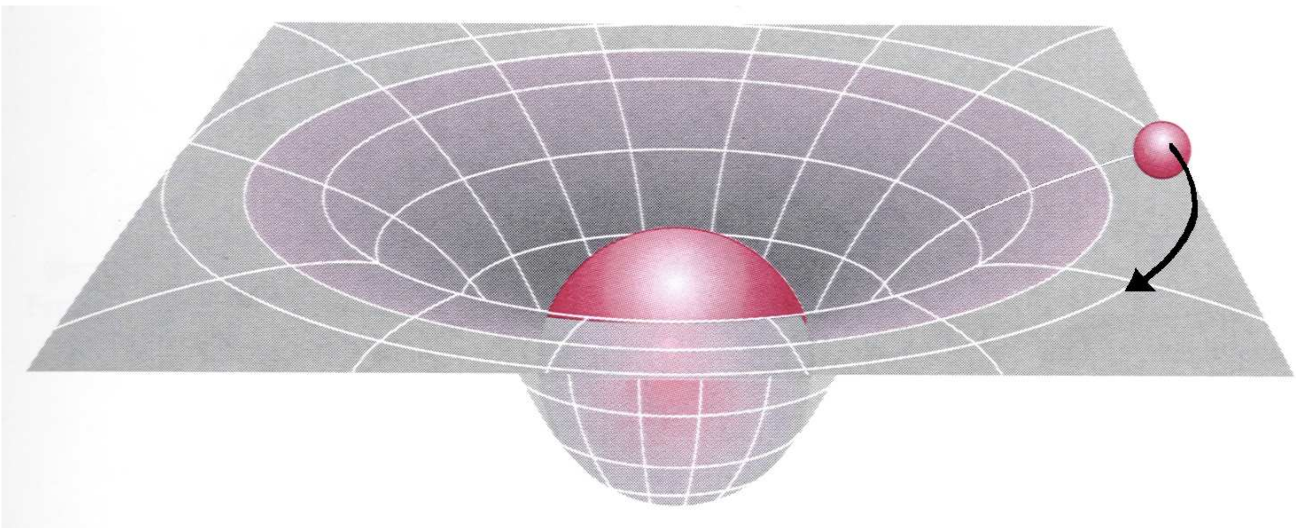

Figure 16: In General Relativity, the warping of the geometry of space-time due to mass distributions accounts for the effects of "gravitational attraction".

This is not simply an alternative but equivalent way of looking at gravity. It would not be such a disturbing idea if that were so ! It is easy to see that dramatic new "gravitational" effects may be predicted.

- Because mass distributions warp space-time, a photon, which also has to travel along a geodesic (locally straight line in the warped space), will also be affected by the mass distribution. Thus General Relativity predicts that photons are subject to gravitational attraction ! Note that the classical theory
$$F = \frac{Gm_1m_2}{r^2} \tag{62}$$
did not predict this for the photon ($m_0 = 0$).
- Very dense matter can warp space so much that nothing, no particles (not even light) can ever escape, once they pass closer than a certain distance, known as the event horizon. Such objects are known as black holes. The density of nuclear matter, when aggregated in amounts equivalent to a large star, is sufficient to realise a black hole.

General Relativity is now widely accepted, following three major experimental verifications :

- The perihelion precession of mercury is beyond that expected by classical theories, but exactly that predicated by General Relativity.

- The gravitational red-shift of light (ie loss of energy ($E_\gamma = hf$) by light as it escapes a gravitating body), as predicted by General Relativity has been quantitatively verified.
- The bending of light in a gravitational field has been verified spectacularly during a lunar eclipse of the sun, and more recently by gravitational lensing.

Black holes have not yet been definitely verified, although there are many strong candidates in the cosmos, and further compelling theoretical evidences.

1.15 The Global Positioning System

1.15.1 Introduction

The GPS system is a practical example of the application of Relativity. Both effects of Special Relativity and General Relativity need to be taken into account. The Special Relativity aspect corrects for the fact that the satellite clocks are moving rather fast with respect to a ground based receiver. The General Relativity aspect corrects for the fact that the satellite clocks are in a different gravitational field. If these effects were not taken into account, a navigational fix based on the GPS constellation would be false after only 2 minutes, and in general errors in global positions would accumulate at a rate of about 10 kilometers each day!

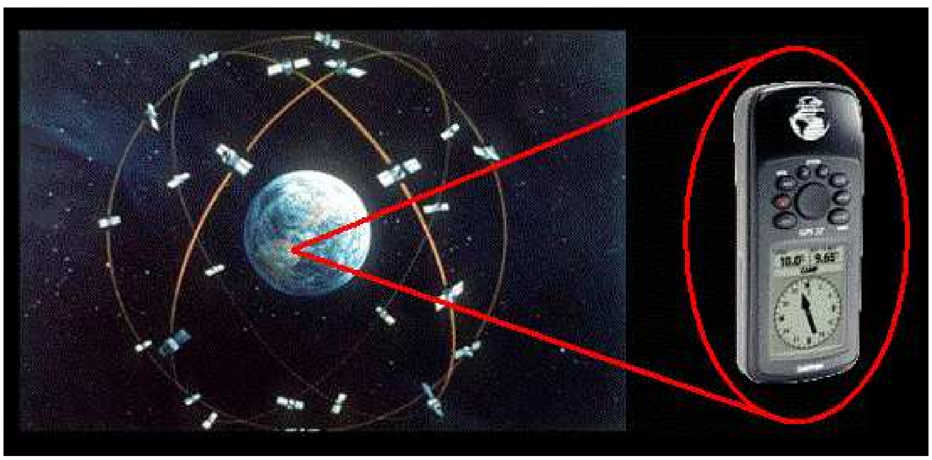

Figure 17: The GPS Satellite system

1.15.2 The GPS system

The Global Positioning System (GPS) consists of a network of 24 satellites in roughly 12-hour orbits, each carrying atomic clocks on board. The orbital radius of the satellites is about four Earth-radii (26,600 km). The orbits are nearly circular, with a typical eccentricity of less than 1%. Orbital inclination to the Earths equator is typically 55 degrees. The satellites have orbital speeds of about 3.9 km/s in a frame centered on the Earth and not rotating with respect to the distant stars. Nominally, the satellites occupy one of six equally spaced orbital planes. Four of them occupy each plane, spread at roughly 90-degree intervals around the Earth in that plane. The precise orbital periods of the satellites are close to 11 hours and 58 minutes so that the ground tracks of the satellites repeat day after day, because the Earth makes one rotation with respect to the stars about every 23 hours and 56 minutes. (Four extra minutes are required for a point on the Earth to return to a position directly under the Sun because the Sun advances about one degree per day with respect to the stars.)

The on-board atomic clocks are good to about 1 nanosecond (ns) in epoch, and about 1 ns/day in rate. Since the speed of light is about 30cm per nanosecond, the system is capable of amazing accuracy in locating anything on Earth or in the near-Earth environment. For example, if the satellite clocks are fully synchronised with ground atomic clocks, and we know the time when a signal is sent from a satellite, then the time delay for that signal to reach a ground receiver immediately reveals the distance (to a potential accuracy of about 30cm) between satellite and ground receiver. By using four satellites to triangulate and determine clock corrections, the position of a receiver at an unknown location can be determined with comparable precision.

1.15.3 Relativistic effects

General Relativity (GR) predicts that clocks in a stronger gravitational field will tick at a slower rate. Special Relativity (SR) predicts that moving clocks will appear to tick slower than non-moving ones. Remarkably, these two effects cancel each other for clocks located at sea level anywhere on Earth. So if a hypothetical clock at Earths north or south pole is used as a reference, a clock at Earths equator would tick slower because of its relative speed due to Earths spin, but faster because of its greater distance from Earths center of mass due to the flattening of the Earth. Because Earths spin rate determines its shape, these two effects are not independent, and it is therefore not entirely coincidental that the effects exactly cancel. The cancellation is not general, however. Clocks at any altitude above sea level do tick faster than clocks at sea level; and clocks on rocket sleds do tick slower than stationary clocks.

For GPS satellites, GR predicts that the atomic clocks at GPS orbital altitudes will tick faster by about 45,900 ns/day because they are in a weaker gravitational field than atomic clocks on Earth's surface. Special Relativity (SR) predicts that atomic clocks moving at GPS orbital speeds will tick slower by about 7,200 ns/day than stationary ground clocks.

1.15.4 Relativistic corrections

The engineers who designed the GPS system took these relativistic effects into account when they designed and deployed the system. One thing they did was to slow down the ticking frequency of the atomic clocks before they were launched so that once they were in their proper orbit stations their clocks would appear to tick at the correct rate as compared to the reference atomic clocks at the GPS ground stations. Further, each GPS receiver has built into it a microcomputer that (among other things) performs the necessary relativistic calculations when determining the user's location.