

# 4 Statistical Mechanics

## 4.1 Introduction

Quantum particles are characterised by wave functions in such a way that they exhibit non-classical behaviour. For example, classical particles are not delocalised, and therefore they are distinguishable. Quantum particles are delocalised and their wave functions will usually overlap. Because of this, a system of identical quantum particles are indistinguishable. This leads to subtle differences in evaluating statistical energy distributions of classical versus quantum particles.

Performing a measurement on the properties of a single quantum particle yields information of a probabilistic nature. Ultimately, in Engineering, we are interested in a system of many particles. Considering macroscopic parameters for the whole system comprised of many particles, we find these are rather stable and predictable, despite the underlying quantum nature. Statistical mechanics describes the macroscopic properties of a system, as illuminated by an understanding of its microscopic sub states. This branch of physics developed initially with a strong link to the behaviour of (classical) gases. Accordingly, we will first consider the statistical mechanics of a gas of classical particles, and develop from there to a system of quantum particles. This system of quantum particles (fermions) will ultimately describe electrons in a semi-conductor (micro-chips).

The classical theories of the Ideal Gas remains important, because at moderate pressures, the separation of the molecules is sufficient to ensure the wave functions of the gas particles does not overlap appreciably. These particles are therefore approximately classical.

## 4.2 Classical Particles : Maxwell-Boltzmann Statistics

Temperature is an example of a macroscopic property that describes the thermal energy of a system of particles. Not all the particles of the system will have the same energy. The total energy  $E$  of the system (if isolated) must remain constant. This total energy  $E$  must be distributed internally, with some probability distribution, among various microscopic energy states  $\epsilon$ . The appropriate probability distribution (stated, not proven) is the Maxwell-Boltzmann distribution function. It gives the probability that a single particle may have energy  $\epsilon$ , if it comes from a system of particles characterised by temperature  $T$ .

$$f_{MB}(\epsilon) = Ae^{-\epsilon/kT} \quad (1)$$

where

$$k = 1.381 \times 10^{-23} \text{J/K} = 8.617 \times 10^{-5} \text{eV/K}$$

is Boltzmann's constant.  $A$  is a normalisation constant so that  $f_{MB}$  can be interpreted as a probability for one atom to occupy the state  $\epsilon$ . Intuitively, higher energy states have an exponentially decreasing chance of being occupied for a given temperature. This distribution may also be thought of as defining the concept of temperature.

### Exercise 1

A cubic meter of atomic hydrogen at  $0^\circ\text{C}$  and at atmospheric pressure contains about  $2.7 \times 10^{25}$  atoms. Find the number of these atoms in their first excited state ( $n = 2$ ) at  $0^\circ\text{C}$  and at  $10,000^\circ\text{C}$

## 4.3 The Energy Distribution for Classical (MB) Particles

The statistical energy distribution for classical particles may now be evaluated. Note that this will be the model for the energy distribution of molecules in an ideal gas. The energy spectrum

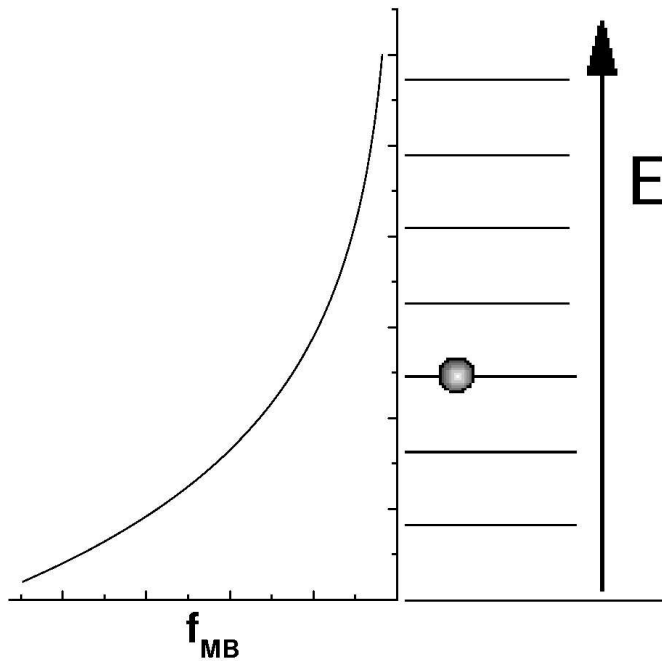


Figure 1: The Maxwell-Boltzmann distribution describing the classical thermal occupancy probability of energy states.

of molecules in a classical gas is continuous. Let  $n(\epsilon)d\epsilon$  describe the number of molecules with energies between  $\epsilon$  and  $\epsilon + d\epsilon$ . Then

$$n(\epsilon)d\epsilon = g(\epsilon)f_{MB}(\epsilon)d\epsilon = g(\epsilon)Ae^{-\epsilon/kT}d\epsilon \quad (2)$$

where  $g(\epsilon)d\epsilon$  is the multiplicity of states with energies between  $\epsilon$  and  $\epsilon + d\epsilon$ . Evaluation of  $g(\epsilon)d\epsilon$  is done in momentum space,

$$p = \sqrt{2m\epsilon} = \sqrt{p_x^2 + p_y^2 + p_z^2} \quad (3)$$

in order to properly count the different momentum components leading to the same energy. The figure below shows that all the momentum vectors leading to the same energy belong in a spherical shell of thickness  $dp$  but with constant radius  $p$  in momentum space. This shell has a volume of

$$\text{shell volume} = 4\pi p^2 dp \quad (4)$$

Therefore

$$\begin{aligned} g(\epsilon)d\epsilon &= g(p)dp = Bp^2 dp \\ &= \sqrt{2}m^{3/2}B\sqrt{\epsilon}d\epsilon \end{aligned} \quad (5)$$

### Exercise 2

Verify the previous equation.

(Hint:  $p^2 = 2m\epsilon$  and  $dp = \frac{m d\epsilon}{\sqrt{2m\epsilon}}$ .)

Substituting equation (5) back into equation (2) we get

$$n(\epsilon)d\epsilon = C\sqrt{\epsilon}e^{-\epsilon/kT}d\epsilon \quad (6)$$

where  $C = 2m^{3/2}AB$  is a constant to be evaluated by the normalisation condition that the total number of molecules is  $N$ .

$$N = \int_0^\infty n(\epsilon)d\epsilon = C \int_0^\infty \sqrt{\epsilon}e^{-\epsilon/kT}d\epsilon. \quad (7)$$

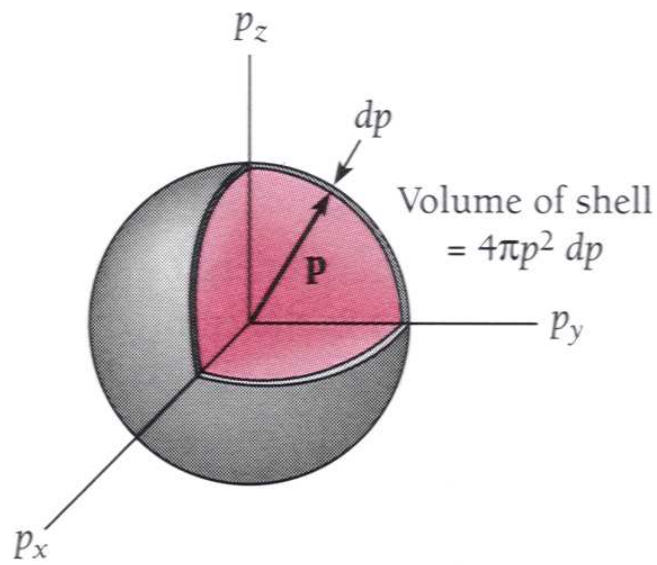


Figure 2: All the momentum vectors leading to the same energy belong in a shell of thickness  $dp$  but with constant radius  $p$  in momentum space.

**Exercise 3**

Show that  $C = \frac{2\pi N}{(\pi kT)^{3/2}}$

(Hint:  $\int_0^\infty \sqrt{x} e^{-ax} = \frac{1}{2a} \sqrt{\frac{\pi}{a}}$ .)

Finally, we find for the number of molecules with energies between  $\epsilon$  and  $\epsilon + d\epsilon$ ,

$$n(\epsilon)d\epsilon = \frac{2\pi N}{(\pi kT)^{3/2}} \sqrt{\epsilon} e^{-\epsilon/kT} d\epsilon \quad (8)$$

This distribution, known as the Maxwellian energy distribution, plotted in figure 3.

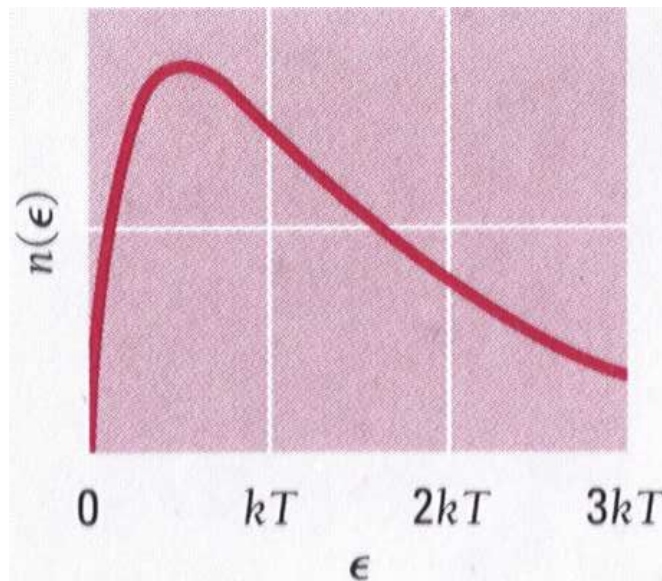


Figure 3: The Maxwellian energy distribution.

The average energy per molecule can be found by distributing the total energy of the system  $E$ ,

over the  $N$  molecules.

$$\begin{aligned} E &= \int_0^\infty \epsilon n(\epsilon) d\epsilon \\ &= \frac{2\pi N}{(\pi kT)^{3/2}} \int_0^\infty \epsilon^{3/2} e^{-\epsilon/kT} d\epsilon \\ &= \frac{3}{2} N kT \end{aligned} \tag{9}$$

#### Exercise 4

Verify this.

(Hint:  $\int_0^\infty x^{3/2} e^{-ax} = \frac{3}{4a^2} \sqrt{\frac{\pi}{a}}$ .)

The average energy per molecule is then

$$\bar{\epsilon} = \frac{3}{2} kT \tag{10}$$

Note that this result is independent of the mass of the molecules. A similar analysis can be performed for the velocity distribution of the gas molecules. The velocity of the molecules will be mass dependent as  $\epsilon = \frac{1}{2}mv^2$ . Lighter molecules will have faster speeds.

#### 4.3.1 Equipartition of Energy

A free monatomic gas particle has three degrees of freedom, corresponding to motion in any of the three orthogonal directions. Each of these motions contributes  $\frac{1}{2}kT$  of energy, so that the net average energy is  $\frac{3}{2}kT$ .

$$\begin{aligned} \frac{1}{2}m\overline{v_x^2} &= \frac{1}{2}m\overline{v_y^2} = \frac{1}{2}m\overline{v_z^2} = \frac{1}{2}kT \\ \frac{1}{2}m\overline{v^2} &= \frac{3}{2}kT \end{aligned}$$

In general, one counts  $\frac{1}{2}kT$  per degree of freedom in the expression of the total energy of the particle. A polyatomic gas molecule will have rotational degrees of freedom as well, and a gas particle confined in a potential may also have degrees of freedom associated with its potential energy.

## 4.4 Indistinguishability of particles and Quantum Statistics

Considering the discussion of §3.8 - §3.10 on systems of indistinguishable particles, we summarize the results.

### 4.4.1 Classical Case - (Distinguishable particles)

$$\Psi_{MI}(\mathbf{r}_1, \mathbf{r}_2) = \psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) \quad (11)$$

or

$$\Psi_{MII}(\mathbf{r}_1, \mathbf{r}_2) = \psi_a(\mathbf{r}_2)\psi_b(\mathbf{r}_1) \quad (12)$$

where  $a$  and  $b$  label two different single particle states.

### 4.4.2 Quantum case - (Indistinguishable particles)

**Bosons :** Symmetric 2-particle wave function

Particles with zero or integral spin (eg. photon  $s = 0$ ).

$$\Psi_B(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}}[\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) + \psi_a(\mathbf{r}_2)\psi_b(\mathbf{r}_1)] \quad (13)$$

**Fermions :** Anti-symmetric 2-particle wave function

Particles with half-integral spin (eg. electron  $s = \frac{1}{2}$ ).

$$\Psi_F(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}}[\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) - \psi_a(\mathbf{r}_2)\psi_b(\mathbf{r}_1)] \quad (14)$$

Note that on insisting the particles are in the same state  $a = b$ , we find for the probability density : -

**Classical particles :**

$$|\Psi_M|^2 = |\psi_a(\mathbf{r}_1)\psi_a(\mathbf{r}_2)|^2 \quad (15)$$

**Bosons :**

$$|\Psi_B|^2 = 2|\psi_a(\mathbf{r}_1)\psi_a(\mathbf{r}_2)|^2 = 2|\Psi_M|^2 \quad (16)$$

**Fermions :**

$$|\Psi_F|^2 = 0 \quad (17)$$

### Exercise 5

Verify the last three results.

Therefore :

1. In a system of fermions, no two particles can be in the same state. (Pauli's exclusion principle).
2. In a system of bosons, the presence of a particle in a certain quantum state increases the probability of another particle to be found in that state. When all bosons accumulate together in the ground state, we have a *Bose Condensate*.
  - (a) This has been observed with superfluid behaviour of liquid helium. (Note : liquid helium  $^4\text{He}$ , not liquid helium  $^3\text{He}$ .) See figure 4.
  - (b) The phenomenon of *superconductivity*, where a conductor has effectively zero resistance below a certain temperature  $T_c$ . Below this temperature,  $T_c$  we have a transition from a system single electrons ( $s = \frac{1}{2} \implies$  fermions) into paired electrons or Cooper pairs ( $s = 0 \implies$  bosons). The scattering of electron-pairs (bosons) is much less than single electrons, as two bosons can occupy the same position. See figure 5.
  - (c) Macroscopic systems of bose condensates have recently been achieved in super-cooled ion-traps. See figure 6.

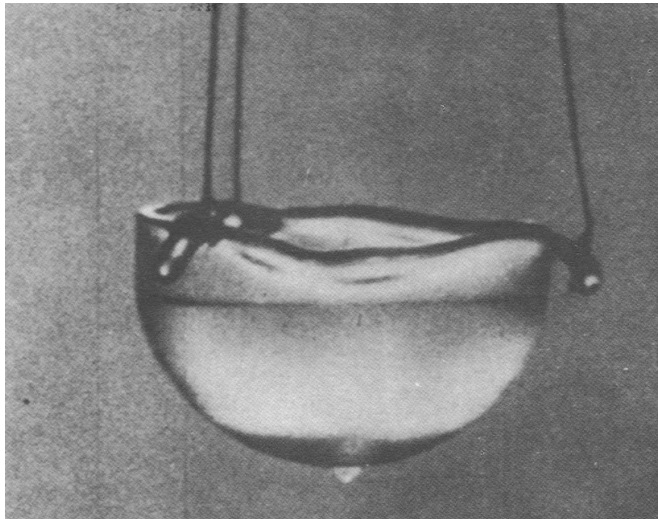


Figure 4: Liquid helium below the lambda point (temperature at which all the helium atoms condense into the same lowest energy state) “siphons” itself out of its container.

## 4.5 Boson Statistics

The appropriate probability distribution for the occupation of energy state  $\epsilon$  (stated, not proven) is the Bose-Einstein distribution function.

$$f_{BE}(\epsilon) = \frac{1}{e^{\epsilon/kT} - 1} \quad (18)$$

### Exercise 6

Show that you can recover the Maxwell-Boltzmann distribution  $f_{MB}$  for large energies. The probability distribution for the occupation of energy state  $\epsilon$  for bosons is always greater than for classical particles, as expected from equation (16).

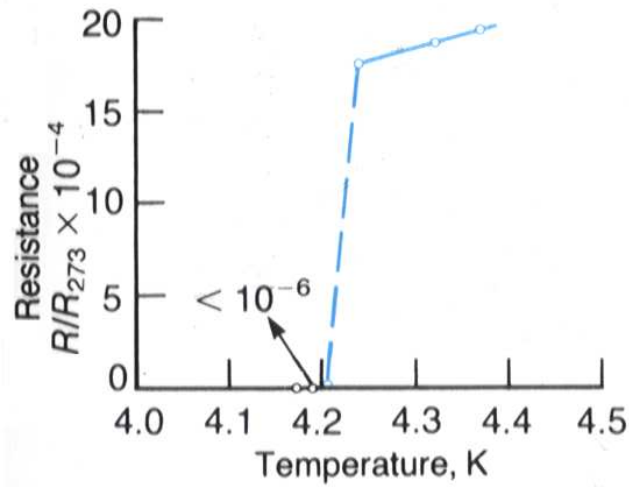


Figure 5: The resistance of some metals drop dramatically below the critical temperature  $T_c$  where Cooper pairing of the electrons occur (so that the charge carriers are now bosons).

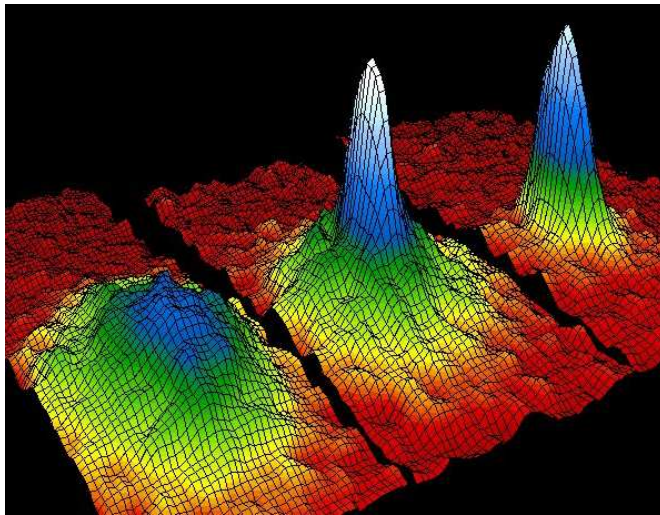


Figure 6: False-color images display the velocity distribution of the cloud of rubidium atoms at (a) just before the appearance of the Bose-Einstein condensate, (b) just after the appearance of the condensate and (c) after further evaporation left a sample of nearly pure condensate. The field of view of each frame is 200 x 270 micrometers. The color corresponds to the number of atoms at each velocity, with red being the fewest and white being the most. Areas appearing white and light blue indicate lower velocities.

## 4.6 The Energy Distribution for Quantum (Bosonic) Particles

In analogy with the case of the ideal gas (section §4.3), the number of photons  $n(\epsilon)d\epsilon$  with energies from  $\epsilon$  to  $\epsilon + d\epsilon$

$$n(\epsilon)d\epsilon = g(\epsilon)f_{BE}(\epsilon)d\epsilon = g(\epsilon)\frac{1}{e^{\epsilon/kT} - 1}d\epsilon \quad (19)$$

where we are using the appropriate Bose-Einstein distribution function for the probability of occupancy of energy state  $\epsilon$ .

We now have to determine the multiplicity of energy states from  $\epsilon$  to  $\epsilon + d\epsilon$ . Considering the photons as quantum particles trapped in a 3-D cavity (generalisation of the particle in a 1-D box of section §2.7.1) we find there is a multiplicity of states given by :-

$$\frac{j_x\pi x}{L} = k_x x \quad j_x = 0, 1, 2, 3, \dots \quad (20)$$

and similarly for the other directions. So, for standing waves in a cubic cavity, we have

$$\begin{aligned} (j_x^2 + j_y^2 + j_z^2) &= \left(\frac{L}{\pi}\right)^2 (k_x^2 + k_y^2 + k_z^2) \\ &= \left(\frac{L}{\pi}\right)^2 k^2 \\ &= \left(\frac{2L}{\lambda}\right)^2 \quad j_x, j_y, j_z = 0, 1, 2, 3, \dots \end{aligned} \quad (21)$$

Now we can consider  $\mathbf{j}$  as a vector in  $j$ -space, so that  $\mathbf{j} \equiv (j_x, j_y, j_z)$  with  $|\mathbf{j}| = j = \sqrt{j_x^2 + j_y^2 + j_z^2}$ . The multiplicity of photons in the cavity with the same wavelength corresponds to the multiplicity of points  $(j_x, j_y, j_z)$  in  $j$ -space which lead to the same magnitude of  $j$ . This will be given by a fraction equal to  $\frac{1}{8}$  of the volume of a spherical shell in  $j$ -space of radius  $j$ . The fraction of  $\frac{1}{8}$  arises as we can only consider positive values of  $j_x, j_y$  and  $j_z$ .

$$g(j)dj = 2 \cdot \frac{1}{8} \cdot 4\pi j^2 dj = \pi j^2 dj \quad (22)$$

The extra factor of 2 in the expression above accounts for the fact that each photon may have two different directions of polarisation (vertical and horizontal). This expression must be converted to a function of frequency  $\nu$  rather than quantum number  $j$ .

$$g(\nu)d\nu = \frac{8\pi V\nu^2}{c^3}d\nu \quad (23)$$

### Exercise 7

Verify the last equation.

(Hint :  $j = \frac{2L}{\lambda} = \frac{2L\nu}{c}$ ,  $dj = \frac{2L\nu}{c}d\nu$ ,  $L^3 = V$ .)

The number distribution of photons in each energy state is the product of the state multiplicity and the probability distribution of the state occupancy

$$\begin{aligned} n(\epsilon)d\epsilon &= g(\epsilon)f_{BE}(\epsilon)d\epsilon \\ &= g(\nu)f_{BE}(\nu)d\nu \\ &= \frac{8\pi V\nu^2}{c^3} \frac{1}{e^{\epsilon/kT} - 1}d\nu. \end{aligned} \quad (24)$$



## 4.7 Black-body radiation and Planck's Radiation Law

An analogy is made between radiation emitted from a perfect black-body and a radiation emitted from a cavity containing radiation which is emitted through a hole in the cavity. Photons are bosons, as the spin of the photon is  $s = 1$ . We are now in a position to understand the quantum behaviour of black body radiation. This is one of the phenomena that lead to the discovery of quantum mechanics. You will recall from section §2.1 that we would need to derive the formula for the spectral energy density distribution (emission spectrum) of black body radiation as :-

$$u(\nu, T) d\nu = \frac{8\pi h}{c^3} \frac{\nu^3 d\nu}{e^{h\nu/kT} - 1}. \quad (25)$$

This last formula is the energy emitted per unit frequency interval per unit volume. The spectral energy density distribution is obtained by weighting the number distribution of the photons in each energy state with the actual photon energy.

$$u(\nu)d\nu = \frac{h\nu n(\nu)d\nu}{V} = \frac{8\pi h}{c^3} \frac{\nu^3 d\nu}{e^{\epsilon/kT} - 1} \quad (26)$$

(noting  $\epsilon = h\nu$ ), which is in fact the Planck radiation formula. It is now clear that the black body radiation spectra is not only a result of quantisation, this is only one aspect of quantum physics. One must also account for the full quantum nature of photons. That is, they are quantum particles which are delocalised and therefore indistinguishable. Accordingly, a system of many photons requires non-classical statistical treatment. The Raleigh-Jeans attempt to predict the spectral energy density distribution for black body radiation failed as it used the classical (Maxwell-Boltzmann) treatment of radiation.

### Exercise 8

How many photons are present in a  $1.00 \text{ cm}^3$  cavity of radiation in thermal equilibrium at  $1000\text{K}$ ? What is their average energy.

The Greenhouse Effect, which is currently much debated, may be understood in terms of black body radiation. Radiation reaching the earth from the sun is well modelled by emanations from a black body with a temperature of  $5250 \text{ K}$  (the solar surface temperature). The spectral energy density distribution of solar radiation therefore peaks at about  $\lambda_{\odot} = 550 \text{ nm}$ . This energy is in the visible region of the spectrum, where the atmosphere is transparent. It is therefore absorbed by the earth. The average surface temperature of the earth is about  $290 \text{ K}$ . The spectral energy density distribution of terrestrial radiation therefore peaks at about  $\lambda_{\text{o}} = 10 \mu\text{m}$ . The outgoing radiation is therefore frequency shifted with respect to the incoming radiation. The total radiant energy received by the earth should balance the energy it re-radiates. However, when the atmosphere has high concentrations of "greenhouse gases" ( $\text{CO}_2$  and  $\text{H}_2\text{O}$ ), it then becomes more opaque to the outgoing radiation. The outgoing radiation is therefore re-absorbed by the atmosphere, altering the energy balance and warming up the planet. A new equilibrium will be established with a slightly higher terrestrial temperature. Figures 7 and 8 illustrate the process.

## 4.8 Fermion Statistics

The appropriate probability distribution for the occupation of energy state  $\epsilon$  (stated, not proven) is the Fermi-Dirac distribution function.

$$f_{FD}(\epsilon) = \frac{1}{e^{(\epsilon - \epsilon_F)/kT} + 1} \quad (27)$$

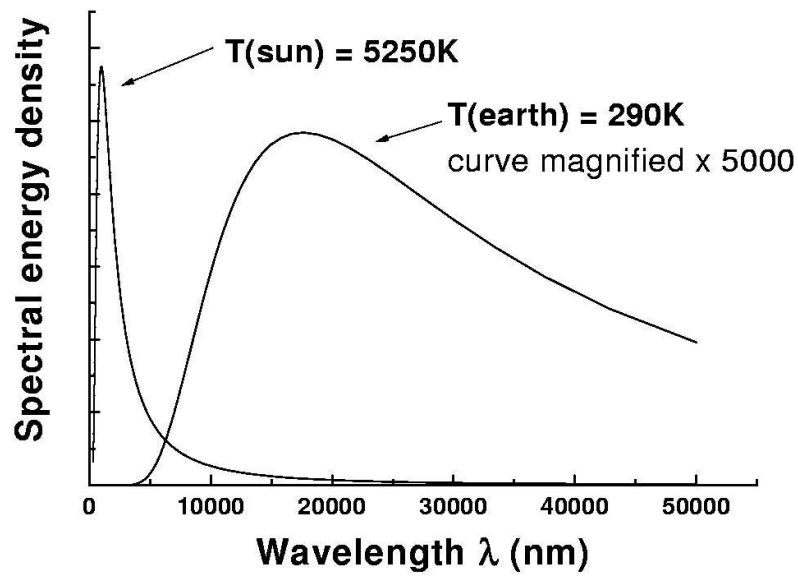


Figure 7: Blackbody radiation spectra emitted from the sun and from the earth.

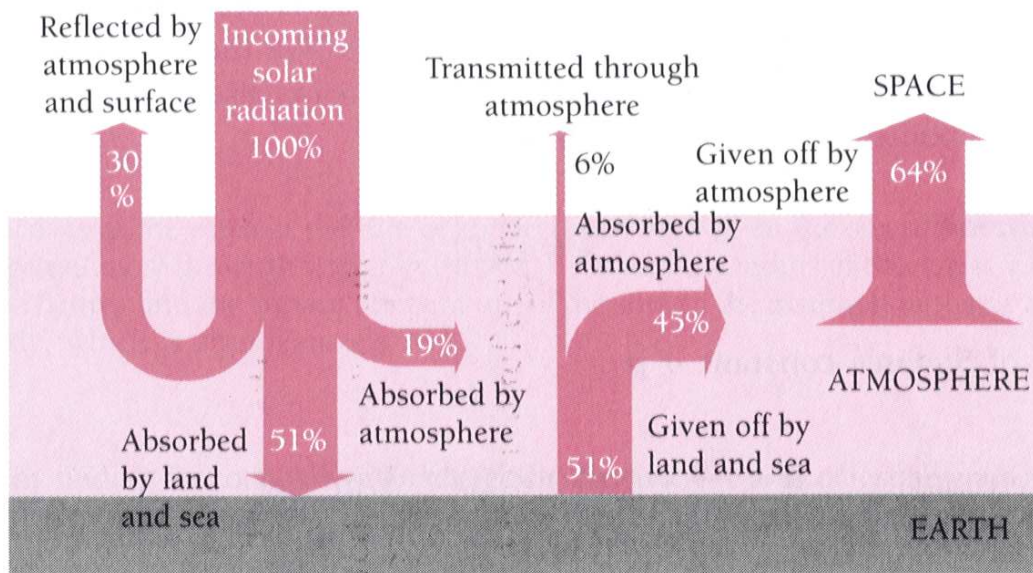


Figure 8: Diagram showing the incoming and outgoing radiation to and from the earth, and its absorption.

This distribution function takes into account that no two fermions may have the same wave function (occupy the same state). In a system consisting of  $N$  electrons at zero temperature, all available states have to be filled sequentially, starting from the lowest energy state. The energy level of the state to which the last of the  $N$  electrons is allocated is known as the Fermi energy level,  $\epsilon_F$ . At zero temperature, we would therefore have the distribution given at the left of figure 9. As we proceed to higher temperatures, the electrons near the Fermi energy level,  $\epsilon_F$ , are excited to an energy level beyond  $\epsilon_F$ , leading to the distribution given at the middle of figure 9. At still higher temperatures, more electrons are excited across the Fermi energy level,  $\epsilon_F$ , leading to the distribution given at the right of figure 9.

### Exercise 7

Explain why it is unlikely that electrons near the ground state could be excited.

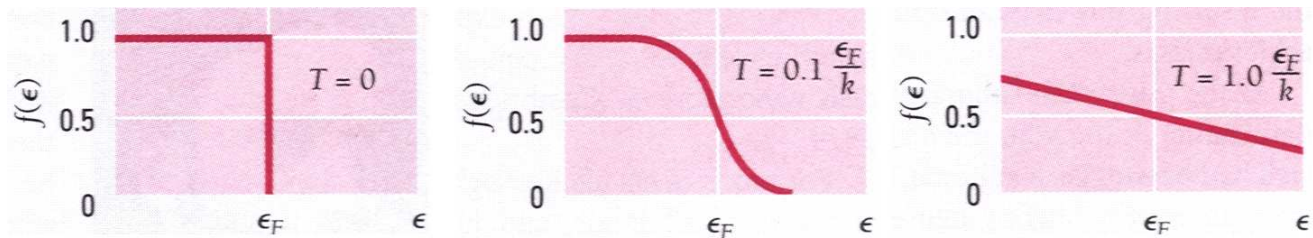


Figure 9: The distribution function for fermions at three different temperatures.

### Exercise 8

Show that you can recover the Maxwell-Boltzmann distribution  $f_{MB}$  for large energies. The probability distribution for the occupation of energy state  $\epsilon$  for fermions is always less than for classical particles, as expected from the Pauli Exclusion Principle.

The three distribution functions

**Maxwell-Boltzmann** for classical particles

**Bose-Einstein** for quantum particles of the boson type

**Fermi-Dirac** for quantum particles of the fermion type

may be compared as in figure 10. The characteristics of the three distribution functions are tabulated in figure 11.

## 4.9 The Energy Distribution for Quantum (Fermionic) Particles

As before, our goal is to find the number of fermions (for example, electrons)  $n(\epsilon)d\epsilon$  with energies from  $\epsilon$  to  $\epsilon + d\epsilon$

$$n(\epsilon)d\epsilon = g(\epsilon)f_{FD}(\epsilon)d\epsilon = g(\epsilon)\frac{1}{e^{(\epsilon-\epsilon_F)/kT} - 1}d\epsilon \quad (28)$$

where we are using the appropriate Fermi-Dirac distribution function for the probability of occupancy of energy state  $\epsilon$ .

We now have to determine (as in section §4.7) the multiplicity of energy states from  $\epsilon$  to  $\epsilon + d\epsilon$ . In direct analogy with the case of photons (bosons) trapped in a 3-D cavity (generalisation of the particle in a box of section §2.7.1) we find there is a multiplicity of states given by

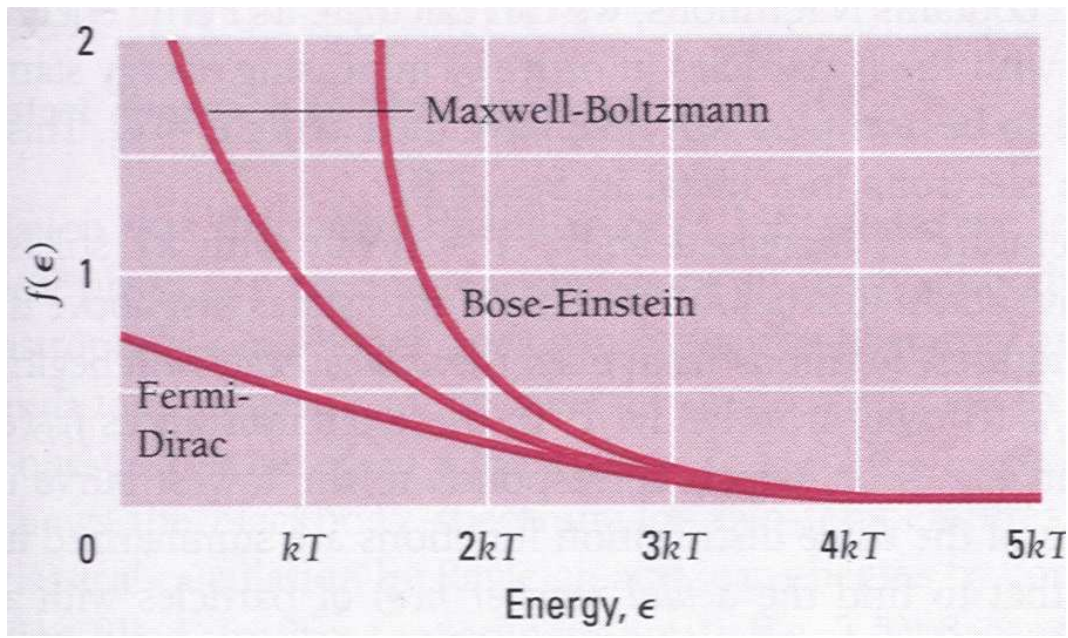


Figure 10: The three distribution functions giving the probability of the occupancy of the energy states,  $\epsilon$ , as a function of temperature.

$$g(j)dj = \pi j^2 dj \quad (29)$$

The analogy is indeed accurate as there are two possible spin states of the electron ( $m_s = \pm\frac{1}{2}$ ) just as there are two possible polarisation states for the photon (vertical and horizontal).

To proceed, we note that

$$j = 2L/\lambda \quad (30)$$

and  $\lambda$  will be given by the de Broglie wavelength

$$\lambda = h/p. \quad (31)$$

As the fermions (for example, electrons in the metal) have non-relativistic velocities,  $p = \sqrt{2m\epsilon}$ , so

$$j = \frac{2L}{\lambda} = \frac{2Lp}{h} = \frac{2L\sqrt{2m\epsilon}}{h} \quad (32)$$

and

$$dj = \frac{L\sqrt{2m}}{h} \frac{d\epsilon}{\epsilon}. \quad (33)$$

Finally we get the desired result,

$$g(\epsilon)d\epsilon = \frac{8\sqrt{2}\pi L^3 m^{3/2}}{h^3} \sqrt{\epsilon} d\epsilon. \quad (34)$$

The number of fermions in a Fermi gas (for example, electrons in a metal) that have energies from  $\epsilon$  to  $\epsilon + d\epsilon$  is

$$\begin{aligned} n(\epsilon)d\epsilon &= g(\epsilon)f_{FD}(\epsilon)d\epsilon \\ &= \frac{8\sqrt{2}\pi L^3 m^{3/2}}{h^3} \frac{\sqrt{\epsilon}d\epsilon}{e^{(\epsilon-\epsilon_F)/kT} + 1} \end{aligned} \quad (35)$$

This distribution is reproduced in figure 12.

	Maxwell-Boltzmann	Bose-Einstein	Fermi-Dirac
Applies to systems of	Identical, distinguishable particles	Identical, indistinguishable particles that do not obey exclusion principle	Identical, indistinguishable particles that obey exclusion principle
Category of particles	Classical	Bosons	Fermions
Properties of particles	Any spin, particles far enough apart so wave functions do not overlap	Spin 0, 1, 2, . . . ; wave functions are symmetric to interchange of particle labels	Spin $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$ ; wave functions are antisymmetric to interchange of particle labels
Examples	Molecules of a gas	Photons in a cavity; phonons in a solid; liquid helium at low temperatures	Free electrons in a metal; electrons in a star whose atoms have collapsed (white) dwarf stars)
Distribution function (number of particles in each state of energy $\epsilon$ at the temperature $T$ )	$f_{MB}(\epsilon) = Ae^{-\epsilon/kT}$	$f_{BE}(\epsilon) = \frac{1}{e^{\alpha}e^{\epsilon/kT} - 1}$	$f_{FD}(\epsilon) = \frac{1}{e^{(\epsilon-\epsilon_F)/kT} + 1}$
Properties of distribution	No limit to number of particles per state	No limit to number of particles per state; more particles per state than $f_{MB}$ at low energies; approaches $f_{MB}$ at high energies	Never more than 1 particle per state; fewer particles per state than $f_{MB}$ at low energies; approaches $f_{MB}$ at high energies

Figure 11: Tabulated characteristics of the three distribution functions.

## 4.10 Electrons in a metal

A simple model of a metal asserts that the electrons fill the metal just as a gas of quantum fermion particles would fill a three-dimensional box. Once again, the “particle in a box” model is used. The volume of the metal defines the dimensions of the box. The attractive potential of the metallic “cores” are approximated by a square well in three dimensions.

This simple model, based on the first quantum mechanical system studied (the particle in a box) is still considered a very effective model of a metal today. Electrons are described by standing waves delocalised in a box of corresponding to the volume of the metal, occupying energy states according to the Fermi-Dirac distribution. Accurate predictions of electron transport, electron contribution to the specific heat of the metal, and electron contribution to magnetism in a metal can be made from this model (beyond the scope of this course).

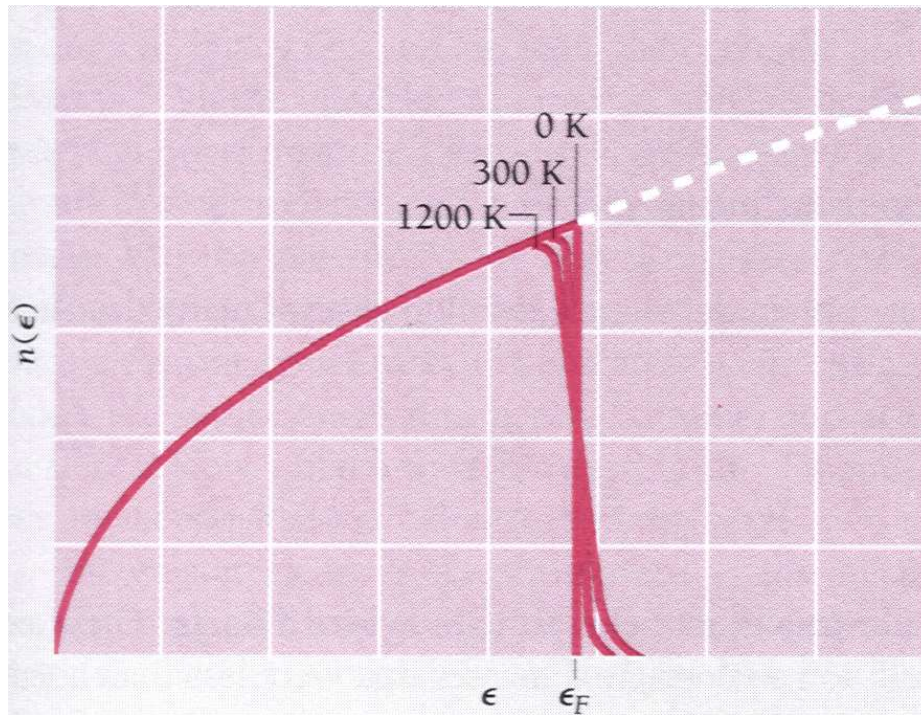


Figure 12: The distribution of electron energies in a metal at various temperatures.

The Fermi energy,  $\epsilon_F$ , may be found by allocating the  $N$  electrons sequentially to the available states at  $T = 0$  from energies  $\epsilon = 0$  to  $\epsilon = \epsilon_F$ .

$$\begin{aligned}
 N &= \int_0^{\epsilon_F} g(\epsilon) d\epsilon & (36) \\
 &= \frac{8\sqrt{2}\pi V m^{3/2}}{h^3} \int_0^{\epsilon_F} \sqrt{\epsilon} d\epsilon \\
 &= \frac{16\sqrt{2}\pi V m^{3/2}}{3h^3} \epsilon_F^{3/2}
 \end{aligned}$$

The Fermi energy is therefore

$$\epsilon_F = \frac{h^2}{2m} \left( \frac{3N}{8\pi V} \right)^{2/3}. \quad (37)$$

The quantity  $N/V$  is the density of free electrons.

## Postscript

The Indian scientist, Chandrasekar, received the Nobel prize for developing a model predicting the fate of stars, once their nuclear fuel was exhausted (figure 13 ). The model balanced the force of gravity (attempting to compress the material of the star) with the Pauli pressure preventing the fermions (electrons) in the star from all being forced into the same state. The Pauli pressure was modelled by treating the electrons in the star as a Fermi gas. As the star cooled, it became dark, as the electrons condensed into the ground state given by the Fermi distribution at  $T = 0$ . In this state, only very few electrons can radiate, as most low lying energy states are already filled, reducing the possibility of radiative transitions. Chandrasekar's model could also show, that for stellar masses greater than  $1.4 M_{sun}$ , the Pauli pressure would be overcome by the gravitational pressure, and the star would collapse catastrophically leading to a violent explosion called a supernova, where many neutrino's would be produced. The remnant of the supernova is a neutron star.

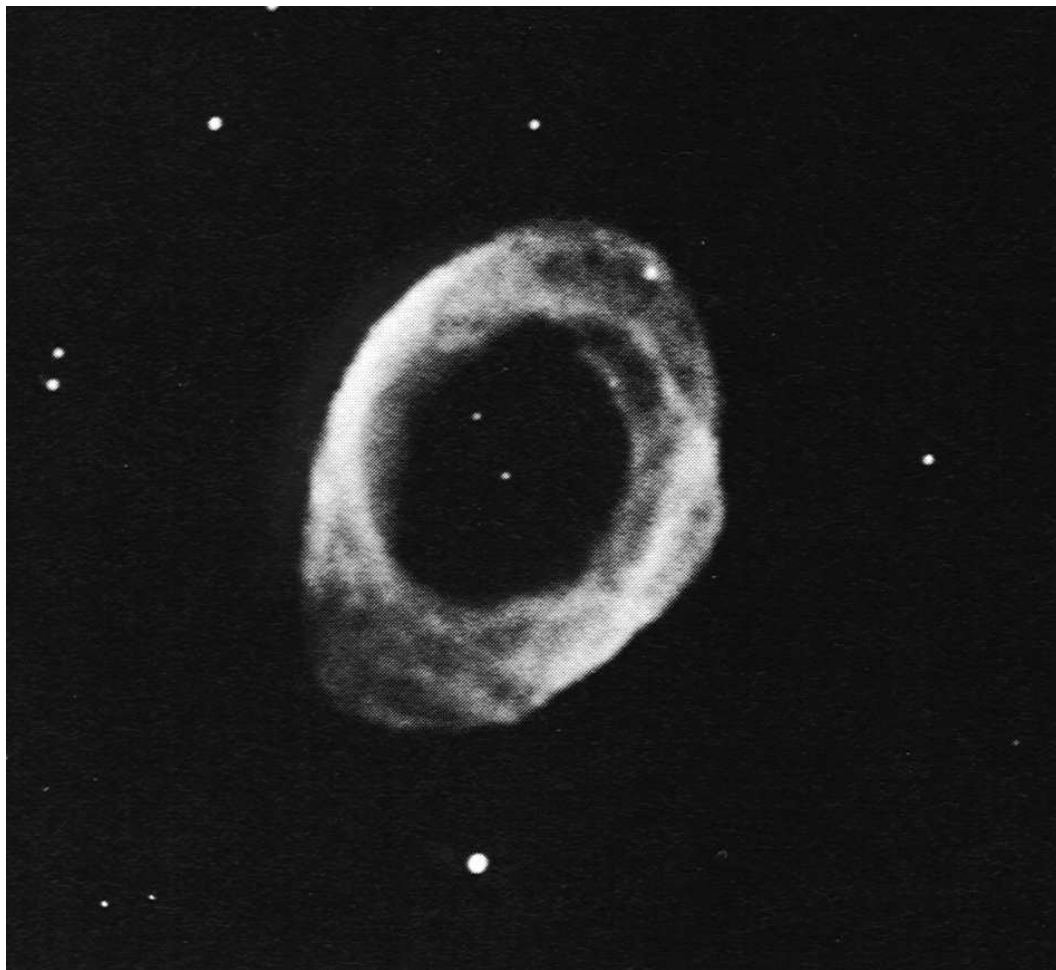


Figure 13: The ring nebula in the constellation of Lyra shows a shell of gas moving away from the star at its centre, which is in the process of becoming a white dwarf.