University of the Witwatersrand
Physics IIE (Engineering) : PHYS284 - 2004
Course Co-ordinator : 1st Semester : Dr SH Connell

Wits Campus, Empire Road, Johannesburg
PO WITS
South Africa
2050

Telephone : (011) 71t-6923
Direct : (011) 717-6928
Fax : (011) 717-6932
e-mail : connell@inkosi.src.wits.ac.za

Quantum Mechanics of Atoms, Statistical Mechanics, Semiconductivity

Class Test 2 — 60 min. (60 marks)

1. (a) The probability of finding an electron, of a given state of the hydrogen atom, at some distance between r and $r + dr$ is given by

$$P(r)dr = r^2 |R_{n,l}(r)|^2 \, dr.$$

How much more likely is a $2p$ electron to be at a distance of $4a_0$ from the nucleus than a distance of a_0 ?
(Use the attached table in figure 2 below of the normalised wave functions for the hydrogen atom.) (8)

(b) Find the ground state energy of the hydrogen atom, $Z = 1$ and $(n = 1, l = 0, m_l = 0)$,

$$E_1 = -\frac{\hbar^2}{2ma_0^2} = -\frac{me^2}{32\pi^2\epsilon_0^2\hbar^2}$$

by substituting the appropriate radial wave function into the radial wave equation

$$\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) + \left[\frac{Ze^2}{4\pi\epsilon_0}\frac{1}{r} - \frac{l(l+1)\hbar^2}{2mr^2} + E\right]R(r) = 0.$$

As an intermediate result, also find an expression for a_0, the Bohr radius,

$$a_0 = \frac{4\pi\epsilon_0\hbar}{me^2} = r_1$$

(Note that for the ground state $l = 0$ and $E = E_1$, and use the attached table in figure 2 below of the normalised wave functions for the hydrogen atom.) (12)

[20]

2. (a) Starting with the wave-functions for two identical non-interacting quantum particles, show that the probability for two fermions to occupy the same quantum state

is zero, and that the probability for two bosons to occupy the same quantum state is twice that for two classical particles to occupy the same state.
(Note : $\Psi_{\frac{B}{F}}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}}[\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) \pm \psi_a(\mathbf{r}_2)\psi_b(\mathbf{r}_1)]$.) (6)

(b) Use this result to state and explain the relative pressure exerted by similar gases of classical molecules, bosons, or fermions for the same temperature. (6)

(c) Sketch the Fermi-Dirac distribution (see question 5a) at low but finite temperature and explain why only electrons near the Fermi-level are expected to participate in transport properties. (5)

(d) Show for this distribution (see question 5a) that if the average occupancy of a state of energy $\epsilon_F + \Delta\epsilon$ is f_1 at any temperature, then the average occupancy of a state of energy $\epsilon_F - \Delta\epsilon$ is $1 - f_1$.
(7)

[25]

3. (a) Classify the materials (a) to (f) below as metals, insulators or semi-conductors specifying the dopant type as well where necessary.

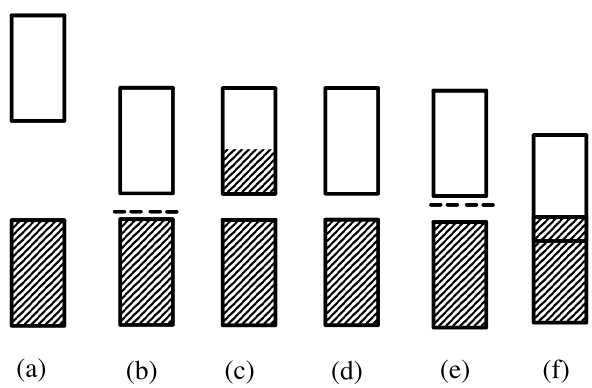

Figure 1: Various materials for Question 6(a).

(6)

(b) In a semi-conductor, the energy of a band is given by $E = Ak^2 + Bk^3$. At what value of k and $E(k)$ will the electron have infinite mass ? (9)

[15]

Total Marks [60]

n	l	m_l	$\Phi(\phi)$	$\Theta(\theta)$	$R(r)$	$\psi(r, \theta, \phi)$
1	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{a_0^{3/2}} e^{-r/a_0}$	$\frac{1}{\sqrt{\pi}\, a_0^{3/2}} e^{-r/a_0}$
2	0	0	$\frac{1}{\sqrt{2\pi}}^{*}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2\sqrt{2}\, a_0^{3/2}} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\, a_0^{3/2}} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$
2	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2} \cos \theta$	$\frac{1}{2\sqrt{6}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \cos \theta$
2	1	± 1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{3}}{2} \sin \theta$	$\frac{1}{2\sqrt{6}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{8\sqrt{\pi}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \sin \theta\, e^{\pm i\phi}$
3	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{81\sqrt{3}\, a_0^{3/2}} \left(27 - 18\frac{r}{a_0} + 2\frac{r^2}{a_0^2}\right) e^{-r/3a_0}$	$\frac{1}{81\sqrt{3\pi}\, a_0^{3/2}} \left(27 - 18\frac{r}{a_0} + 2\frac{r^2}{a_0^2}\right) e^{-r/3a_0}$
3	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2} \cos \theta$	$\frac{4}{81\sqrt{6}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0}$	$\frac{\sqrt{2}}{81\sqrt{\pi}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0} \cos \theta$
3	1	± 1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{3}}{2} \sin \theta$	$\frac{4}{81\sqrt{6}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0} \sin \theta\, e^{\pm i\phi}$
3	2	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{10}}{4} (3 \cos^2 \theta - 1)$	$\frac{4}{81\sqrt{30}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{1}{81\sqrt{6\pi}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0} (3 \cos^2 \theta - 1)$
3	2	± 1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{15}}{2} \sin \theta \cos \theta$	$\frac{4}{81\sqrt{30}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0} \sin \theta \cos \theta\, e^{\pm i\phi}$
3	2	± 2	$\frac{1}{\sqrt{2\pi}} e^{\pm 2i\phi}$	$\frac{\sqrt{15}}{4} \sin^2 \theta$	$\frac{4}{81\sqrt{30}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{1}{162\sqrt{\pi}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0} \sin^2 \theta\, e^{\pm 2i\phi}$

*The quantity $a_0 = 4\pi\epsilon_0 \hbar^2/me^2 = 5.292 \times 10^{-11}$ m is equal to the radius of the innermost Bohr orbit.

Figure 2: Normalised wave functions of the hydrogen atom for $n = 1, 2$ and 3.