University of the Witwatersrand
Physics IIE (Engineering) : PHYS2007 - 2007
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Introduction to QM, The H-atom, Statistical Mechanics

Class Test 2 — 60 + 10 min. (60 marks)

1. (a) With regard to the Scanning Tunneling Microscope :

   i. Explain the principle of operation by referring to a particle energy drawing showing clearly the three regions corresponding to the sample, the tip and the air-gap in an idealised barrier penetration problem. (4)

   ii. The transmission coefficient of quantum particles through the barrier is given by

   $$T = e^{-2\frac{\sqrt{2m(U-E)}}{\hbar}L}.$$

   Indicate the meaning of the symbols $(U-E)$, L and m. (3)

   iii. For a specific tunneling microscope, we can write

   $$T = e^{-2CL}$$

   where $C = 10.0\text{nm}^{-1}$. Suppose the electronic detection system is sensitive to electrical current changes of 0.1%. What surface height sensitivity does this correspond to. (4)

   (b) The tunneling process also represents a physics limit for the miniaturisation of features on a chip. This could be either tunneling between neighbouring wires or across the gate of a transistor. The corresponding classical process is thermally stimulated over-barrier hopping.

   i. Imagine that the quantum process should not be more likely than the classical process, so that the limiting case is when they are equal. This will allow you to estimate a value for the minimum feature size L of a conventional chip. Estimate realistic values for $(U-E)$ and kT and explain your choice. (4)

   ii. Perform the calculation to find L. (5)

Total for Question 2 [20]

2 (a) The probability of finding an electron, of a given state of the hydrogen atom, at some distance between r and $r + dr$ is given by

$$P(r)dr = r^2|R_{n,l}(r)|^2\, dr.$$

How much more likely is a $2p$ electron to be at a distance of $4a_0$ from the nucleus than a distance of a_0 ?
(Use the attached table in figure 1 below of the normalised wave functions for the hydrogen atom.) (8)

(b) Find the ground state energy of the hydrogen atom, $Z = 1$ and $(n = 1, l = 0, m_l = 0)$,

$$E_1 = -\frac{\hbar^2}{2ma_0^2} = -\frac{me^2}{32\pi^2\epsilon_0^2\hbar^2}$$

by substituting the appropriate radial wave function into the radial wave equation

$$\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) + \left[\frac{Ze^2}{4\pi\epsilon_0}\frac{1}{r} - \frac{l(l+1)\hbar^2}{2mr^2} + E\right]R(r) = 0.$$

As an intermediate result, also find an expression for a_0, the Bohr radius,

$$a_0 = \frac{4\pi\epsilon_0\hbar}{me^2} = r_1$$

(Note that for the ground state $l = 0$ and $E = E_1$, and use the attached table in figure 1 below of the normalised wave functions for the hydrogen atom.) (12)

[20]

3 (a) The number of fermions in a fermi gas (for example, electrons in a metal) that have energies from ϵ to $\epsilon + d\epsilon$ is

$$\begin{aligned} n(\epsilon)d\epsilon &= g(\epsilon)f_{FD}(\epsilon)d\epsilon \\ &= \frac{8\sqrt{2}\pi L^3 m^{3/2}}{h^3}\frac{\sqrt{\epsilon}d\epsilon}{e^{(\epsilon-\epsilon_F)/kT}+1} \end{aligned}$$

Consider N electrons in the low temperature limit where all the low-lying states are occupied. Show that the energy of the last filled state, known as the fermi energy, is given by

$$\epsilon_F = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3}.$$

(8)

(b) Show that the median energy in a free electron gas at $T = 0$ is $\epsilon_F/2^{2/3} = 0.630\epsilon_F$.
(Hint: The median energy defines the point where the total number of states below this value is equal to the total number of states above this value.) (4)

(c) The density of metallic zinc is 7.13 g/cm^3 and the atomic mass of the zinc atom is 65.4 u. The Fermi energy in zinc metal is 11.0 eV. Work out the effective mass of a delocalised electron in Zinc metal. Express your answer in terms of the free electron mass. Why is there a difference ? (8)

[20]

Total Marks [60]

n	l	m_l	$\Phi(\phi)$	$\Theta(\theta)$	$R(r)$	$\psi(r, \theta, \phi)$
1	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{a_0^{3/2}} e^{-r/a_0}$	$\frac{1}{\sqrt{\pi}\, a_0^{3/2}} e^{-r/a_0}$
2	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2\sqrt{2}\, a_0^{3/2}} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\, a_0^{3/2}} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$
2	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2} \cos\theta$	$\frac{1}{2\sqrt{6}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \cos\theta$
2	1	±1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{3}}{2} \sin\theta$	$\frac{1}{2\sqrt{6}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{8\sqrt{\pi}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \sin\theta\, e^{\pm i\phi}$
3	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{81\sqrt{3}\, a_0^{3/2}} \left(27 - 18\frac{r}{a_0} + 2\frac{r^2}{a_0^2}\right) e^{-r/3a_0}$	$\frac{1}{81\sqrt{3\pi}\, a_0^{3/2}} \left(27 - 18\frac{r}{a_0} + 2\frac{r^2}{a_0^2}\right) e^{-r/3a_0}$
3	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2} \cos\theta$	$\frac{4}{81\sqrt{6}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0}$	$\frac{\sqrt{2}}{81\sqrt{\pi}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0} \cos\theta$
3	1	±1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{3}}{2} \sin\theta$	$\frac{4}{81\sqrt{6}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0} \sin\theta\, e^{\pm i\phi}$
3	2	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{10}}{4} (3\cos^2\theta - 1)$	$\frac{4}{81\sqrt{30}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{1}{81\sqrt{6\pi}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0} (3\cos^2\theta - 1)$
3	2	±1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{15}}{2} \sin\theta\cos\theta$	$\frac{4}{81\sqrt{30}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0} \sin\theta\cos\theta\, e^{\pm i\phi}$
3	2	±2	$\frac{1}{\sqrt{2\pi}} e^{\pm 2i\phi}$	$\frac{\sqrt{15}}{4} \sin^2\theta$	$\frac{4}{81\sqrt{30}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{1}{162\sqrt{\pi}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0} \sin^2\theta\, e^{\pm 2i\phi}$

*The quantity $a_0 = 4\pi\epsilon_0\hbar^2/me^2 = 5.292 \times 10^{-11}$ m is equal to the radius of the innermost Bohr orbit.

Figure 1: Normalised wave functions of the hydrogen atom for $n = 1, 2$ and 3.