University of the Witwatersrand
Physics IIE (Engineering) : PHYS284 : 2002
Examination : June 2002 : Deferred

Instructions: Answer all questions.
Time: 2 hours = 120 minutes
Total Marks: 120 marks

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1. The relativistic expression for kinetic energy is

$$E_k = mc^2 - m_0c^2.$$

(a) Show that this expression reduces to the normal classical expression $\frac{1}{2}m_0v^2$ at low speeds. (6)

(b) Given that the Lorentz Transformation is :

$$\begin{aligned} x' &= \frac{(x - vt)}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{(t - \frac{vx}{c^2})}{\sqrt{1 - v^2/c^2}} \end{aligned}$$

derive the formula for the relativistic length contraction

$$L = L_0\sqrt{1 - v^2/c^2}.$$

(4)

(c) A space shuttle 2 light years away from a space station approaches the space station with a speed of $c/2$ from the point of view of observers on the space station.

i) How long do astronauts on the space station think the shuttle will take to reach them ? (2)

ii) How long do astronauts on the shuttle think they will take to reach the space station ? (3)

iii) If another space shuttle were approaching the space station from the opposite direction, but with the same speed, then what would be the relative velocity of the two shuttles, according to Galilean relativity ? (2)

iv) What would be the actual relative velocity of the two shuttles ? (3)

[20]

2. (a) A linear quantum dot can be modelled as an electron in a box. It can be formed by depositing contacts on a slab of GaAs and then applying a bias voltage in such a way as to create a linear confining potential.

i) Suppose the observed ground-state electron energy is 0.003 eV. How big is the potential boundary ("box") of the quantum dot ?
(Note : for GaAs, $m^* = 0.067 m_e$.) (7)

ii) Show that this result is roughly consistent with Heisenberg's Uncertainty Principle. (4)

(b) With regard to the Scanning Tunneling Microscope :

i) Explain the principle of operation by referring to a particle energy drawing showing clearly the three regions corresponding to the sample, the tip and the air-gap in an idealised barrier penetration problem. (4)

ii) The transmission coefficient of electrons through the barrier in a tunneling microscope is given by

$$T = e^{-2CL}$$

where $C = 10.0\text{nm}^{-1}$. Suppose the electronic detection system is sensitive to electrical current changes of 0.1%. What surface height sensitivity does this correspond to. (5)

[20]

3. (a) The probability of finding the electron in the hydrogen atom ground state, at some distance between r and $r + dr$ is given by

$$P(r)dr = \frac{1}{a_0}\left(\frac{2r}{a_0}\right)^2 e^{-2r/a_0}\, dr$$

where a_0 is the Bohr radius (0.05292 nm). Make a rough plot of $P(r)$ in terms of r/a_0 and calculate the distance from the nucleus where the electron is most likely to be found. (15)

(b) The energy levels of certain kinds of two-electron atoms/ions of atomic number Z may be approximated by

$$E_n(Z) = E_1(H)Z^2 - \frac{E_1(H)(Z-1)^2}{n^2}$$

where the ground state of hydrogen is $E_1(H) = -13.6\text{eV}$.

i) Sketch the two-electron atom/ion under conditions where the above expression could be expected to hold. (5)

ii) Using the full expression for quantised energy levels in hydrogen-like (that is, one-electron) atoms/ions of atomic number Z, describe how each term in the above expression arises. (6)

iii) Why does the accuracy increase when n increases ? (4)

[30]

4. (a) Show that electronic density of states

$$n(\epsilon)d\epsilon = \frac{8\sqrt{2}\pi V m^{3/2}}{h^3} \frac{\sqrt{\epsilon}d\epsilon}{e^{(\epsilon-\epsilon_F)/kT}+1}$$

may be expressed as

$$n(\epsilon)d\epsilon = (3N/2)\epsilon_F^{-3/2}\sqrt{\epsilon}d\epsilon \qquad \text{for} \qquad \epsilon < \epsilon_F$$

at 0 K. (7)

(b) Now show that the total energy of the electron gas at 0 K is

$$E_0 = \frac{3}{5}N\epsilon_F$$

by evaluating

$$E_0 = \int_0^{\epsilon_F} \epsilon n(\epsilon)d\epsilon.$$

and hence arrive at the result

$$\overline{\epsilon}_0 = \frac{3}{5}\epsilon_F$$

Which is typically a few electron volts in magnitude. (6)

(c) If electrons behaved classically, what would the temperature of the metal have to be for a similar average energy to of the electrons to be realised. (6)

(d) With respect to the Fermi-Dirac distribution, show that if the average occupancy of a state of energy $\epsilon_F + \Delta\epsilon$ is f_1 at any temperature, then the average occupancy of a state of energy $\epsilon_F - \Delta\epsilon$ is $1 - f_1$. (6)

[25]

5. (a) In an intrinsic semiconductor, $n(\epsilon)d\epsilon = g(\epsilon)f_{FD}(\epsilon)d\epsilon$, where

$$f_{FD}(\epsilon) = \frac{1}{e^{(\epsilon-\epsilon_F)/kT}+1}$$

and

$$g_e(\epsilon)d\epsilon = \frac{8\sqrt{2}\pi V m_e^{*3/2}}{h^3}\sqrt{\epsilon-\epsilon_g}\,d\epsilon, \qquad g_h(\epsilon)d\epsilon = \frac{8\sqrt{2}\pi V m_h^{*3/2}}{h^3}\sqrt{-\epsilon}\,d\epsilon.$$

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Show that the number of electrons in the conduction band is :

$$n_e = 2V \left(\frac{m_e^* kT}{2\pi \hbar^2} \right)^{3/2} e^{(\epsilon_F - \epsilon_g)/kT}.$$

(Hint : $\int y^{\frac{1}{2}} e^{-y}\, dy = \frac{\pi^{\frac{1}{2}}}{2}$.) (7)

(b) A similar equation can be shown to hold for the number of holes in the valence band :

$$n_h = 2V \left(\frac{m_h^* kT}{2\pi \hbar^2} \right)^{3/2} e^{-\epsilon_F/kT}.$$

Considering this, show

$$\epsilon_F = \epsilon_g/2$$

at room temperature in an intrinsic semiconductor. (6)

(c) Also derive the expression

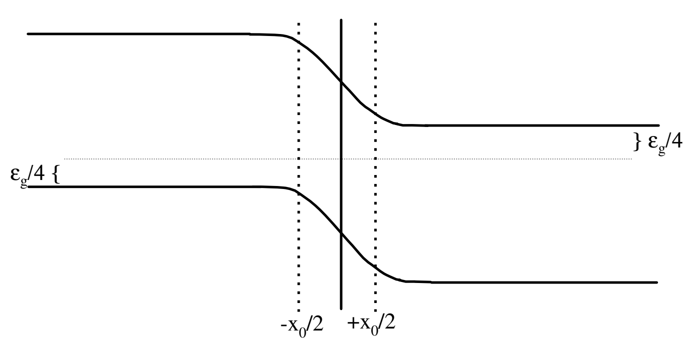
$$n_i = n_0 e^{-\epsilon_g/2kT}$$

for the number of intrinsic carriers and give the explicit form of n_0. Note that in the case of a donor-doped semiconductor, the number of carriers is

$$n_c = n_i e^{(\epsilon_F - \epsilon_i)/kT}.$$

(6)

(d) A semiconductor is characterised by the energy band structure shown in the following figure. Determine the resistivity of the $x_0 > x_0/2$ section of the semiconductor. ($\epsilon_g = 1.12\text{eV}$, $n_i = 1.18 \times 10^{10}\text{cm}^{-3}$, $kT = 0.026\text{eV}$ $\mu = 1350\text{cm}^2\text{V}^{-1}\text{s}^{-1}$, $e = 1.6 \times 10^{-19}\text{C}$.) (6)



[25]

Total Marks **[120]**

n	l	m_l	$\Phi(\phi)$	$\Theta(\theta)$	$R(r)$	$\psi(r, \theta, \phi)$
1	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{a_0^{3/2}} e^{-r/a_0}$	$\frac{1}{\sqrt{\pi}\, a_0^{3/2}} e^{-r/a_0}$
2	0	0	$\frac{1}{\sqrt{2\pi}}$*	$\frac{1}{\sqrt{2}}$	$\frac{1}{2\sqrt{2}\, a_0^{3/2}} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\, a_0^{3/2}} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$
2	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2} \cos\theta$	$\frac{1}{2\sqrt{6}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \cos\theta$
2	1	± 1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{3}}{2} \sin\theta$	$\frac{1}{2\sqrt{6}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{8\sqrt{\pi}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \sin\theta\, e^{\pm i\phi}$
3	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{81\sqrt{3}\, a_0^{3/2}} \left(27 - 18\frac{r}{a_0} + 2\frac{r^2}{a_0^2}\right) e^{-r/3a_0}$	$\frac{1}{81\sqrt{3\pi}\, a_0^{3/2}} \left(27 - 18\frac{r}{a_0} + 2\frac{r^2}{a_0^2}\right) e^{-r/3a_0}$
3	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2} \cos\theta$	$\frac{4}{81\sqrt{6}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0}$	$\frac{\sqrt{2}}{81\sqrt{\pi}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0} \cos\theta$
3	1	± 1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{3}}{2} \sin\theta$	$\frac{4}{81\sqrt{6}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0} \sin\theta\, e^{\pm i\phi}$
3	2	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{10}}{4} (3\cos^2\theta - 1)$	$\frac{4}{81\sqrt{30}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{1}{81\sqrt{6\pi}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0} (3\cos^2\theta - 1)$
3	2	± 1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{15}}{2} \sin\theta\cos\theta$	$\frac{4}{81\sqrt{30}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0} \sin\theta\cos\theta\, e^{\pm i\phi}$
3	2	± 2	$\frac{1}{\sqrt{2\pi}} e^{\pm 2i\phi}$	$\frac{\sqrt{15}}{4} \sin^2\theta$	$\frac{4}{81\sqrt{30}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{1}{162\sqrt{\pi}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0} \sin^2\theta\, e^{\pm 2i\phi}$

*The quantity $a_0 = 4\pi\epsilon_0\hbar^2/me^2 = 5.292 \times 10^{-11}$ m is equal to the radius of the innermost Bohr orbit.

Figure 1: Normalised wave functions of the hydrogen atom for $n = 1, 2$ and 3.