University of the Witwatersrand
Physics IIE (Engineering) : PHYS284 : 2002
**Examination : June 2002**

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**Instructions:** Answer all questions.
Time: 2 hours = 120 minutes
Total Marks: 120 marks

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1. The relativistic expression for kinetic energy is

$$E_k = mc^2 - m_0c^2.$$

   (a) Show that this expression reduces to the normal classical expression $\frac{1}{2}m_0v^2$ at low speeds. (6)

   (b) Sketch the relativistic and classical variation of kinetic energy with velocity. (4)

   (c) What approximate fractional error does one make in using $\frac{1}{2}m_0v^2$ for the kinetic energy of a body if its speed is $v = 0.1c$, and again if its speed is $v = 0.9c$ ? Give fractions to two significant figures.
   (Hint : fractional error = (true value - approximate value)/true value).) (7)

   (d) An electron in a TV tube is accelerated by a potential of 17 kV. Calculate its

      i) total energy,
      ii) $\gamma$, the fractional mass increase,
      iii) velocity.

   Use units of keV for both energies and mass (i.e. $mc^2$), and $c$ for velocity.
   Suppose that you want to design a high resolution monitor,

      iv) Discuss the relativistic effects on your design, and how important they are to consider. (8)

[25]

2. (a) A linear quantum dot can be modelled as an electron in a box. It can be formed by depositing contacts on a slab of GaAs and then applying a bias voltage in such a way as to create a linear confining potential.

      i) Suppose the observed ground-state electron energy is 0.003 eV. How big is the potential boundary ("box") of the quantum dot ?
      (Note : for GaAs, $m^* = 0.067m_e$.) (7)

      ii) Show that this result is roughly consistent with Heisenberg's Uncertainty Principle. (4)

   (b) With regard to the Scanning Tunneling Microscope :

      i) Explain the principle of operation by referring to a particle energy drawing showing clearly the three regions corresponding to the sample, the tip and the air-gap in an idealised barrier penetration problem. (4)

ii) The transmission coefficient of electrons through the barrier in a tunneling microscope is given by

$$T = e^{-2CL}$$

where $C = 10.0\text{nm}^{-1}$. Suppose the electronic detection system is sensitive to electrical current changes of 0.1%. What surface height sensitivity does this correspond to. (5)

[20]

3. (a) The probability of finding an electron, of a given state of the hydrogen atom, at some distance between $r$ and $r + dr$ is given by

$$P(r)dr = r^2 |R_{n,l}(r)|^2 \, dr.$$

How much more likely is a $2p$ electron to be at a distance of $4a_0$ from the nucleus than a distance of $a_0$ ? (Use the attached table in figure 2 below of the normalised wave functions for the hydrogen atom.) (8)

(b) Unsöld's theorem states that for any value of the orbital quantum number $l$, the probability densities summed over all possible values of $m_l$ are angle independent.

i) Verify this for $l = 1$ by showing $\Sigma_{m_l=-l}^{+l} |\Theta_{l,m_l}|^2 |\Phi_{m_l}|^2 = const$ using the attached table in figure 2 below of the normalised wave functions for the hydrogen atom. (6)

ii) What is the physical meaning of this mathematical fact. (2)

iii) Which group of the periodic table contains elements demonstrating Unsöld's theorem. (2)

iv) How does the mathematical form of the probability density illuminate the properties of these elements ? (2)

[20]

4. (a) Show that electronic density of states

$$n(\epsilon)d\epsilon = \frac{8\sqrt{2}\pi V m^{3/2}}{h^3} \frac{\sqrt{\epsilon} d\epsilon}{e^{(\epsilon - \epsilon_F)/kT} + 1}$$

may be expressed as

$$n(\epsilon)d\epsilon = (3N/2)\epsilon_F^{-3/2}\sqrt{\epsilon}d\epsilon \qquad \text{for} \qquad \epsilon < \epsilon_F$$

at 0 K. (6)

(b) Now show that the total energy of the electron gas at 0 K is

$$E_0 = \frac{3}{5}N\epsilon_F$$

by evaluating

$$E_0 = \int_0^{\epsilon_F} \epsilon n(\epsilon) d\epsilon.$$

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and hence arrive at the result

$$\bar{\epsilon}_0 = \frac{3}{5}\epsilon_F$$

Which is typically a few electron volts in magnitude. (6)

(c) If electrons behaved classically, what would the temperature of the metal have to be for a similar average energy to of the electrons to be realised. (6)

(d) With respect to the Fermi-Dirac distribution, show that if the average occupancy of a state of energy $\epsilon_F + \Delta\epsilon$ is $f_1$ at any temperature, then the average occupancy of a state of energy $\epsilon_F - \Delta\epsilon$ is $1 - f_1$. (6)

[24]

5. Silver (Group I) has a molar mass of 108 g/mol, a density of $10.5 \times 10^3$ kg/m$^3$ and a Fermi energy of 5.54 eV. The resistivity is $1.6 \times 10^{-8}\Omega\cdot$m. Show that the mean free path $\lambda$ between collisions of the free electrons is about 50 nm. (6)

[6]

6. (a) Classify the materials (a) to (f) below as metals, insulators or semi-conductors specifying the dopant type as well where necessary.

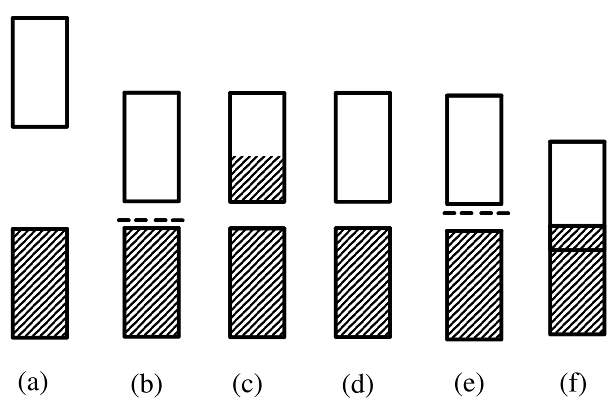


Figure 1: Various materials for Question 6(a).

(6)

(b) By setting the number of hole and electron charge carriers equal for intrinsic material, show that the Fermi-level is approximately in the middle of the band gap. (6)

(c) Now consider doped material. Using the following data,

$$\epsilon_r = 11.7 \text{ for Si}$$
$$m_e^*/m_e = 0.19 \text{ near the bottom of the conduction band in Si}$$

Calculate the ionisation energy of the phosphorous atom in silicon. (5)

(d) Hence explain the location of the donor state in the band gap. (3)

(e) What is the new position of the Fermi level ? (2)

(f) Compare the ionisation energy of the donor state to the energy represented by room temperature and comment on the consequences of this for the charge carrier concentration. (3)

[25]

**Total Marks** **[120]**

| $n$ | $l$ | $m_l$ | $\Phi(\phi)$ | $\Theta(\theta)$ | $R(r)$ | $\psi(r,\theta,\phi)$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | $\frac{1}{\sqrt{2\pi}}$ | $\frac{1}{\sqrt{2}}$ | $\frac{2}{a_0^{3/2}} e^{-r/a_0}$ | $\frac{1}{\sqrt{\pi}\, a_0^{3/2}} e^{-r/a_0}$ |
| 2 | 0 | 0 | $\frac{1}{\sqrt{2\pi}}$* | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2\sqrt{2}\, a_0^{3/2}} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$ | $\frac{1}{4\sqrt{2\pi}\, a_0^{3/2}} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$ |
| 2 | 1 | 0 | $\frac{1}{\sqrt{2\pi}}$ | $\frac{\sqrt{6}}{2} \cos\theta$ | $\frac{1}{2\sqrt{6}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0}$ | $\frac{1}{4\sqrt{2\pi}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \cos\theta$ |
| 2 | 1 | ±1 | $\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$ | $\frac{\sqrt{3}}{2} \sin\theta$ | $\frac{1}{2\sqrt{6}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0}$ | $\frac{1}{8\sqrt{\pi}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \sin\theta\, e^{\pm i\phi}$ |
| 3 | 0 | 0 | $\frac{1}{\sqrt{2\pi}}$ | $\frac{1}{\sqrt{2}}$ | $\frac{2}{81\sqrt{3}\, a_0^{3/2}} \left(27 - 18\frac{r}{a_0} + 2\frac{r^2}{a_0^2}\right) e^{-r/3a_0}$ | $\frac{1}{81\sqrt{3\pi}\, a_0^{3/2}} \left(27 - 18\frac{r}{a_0} + 2\frac{r^2}{a_0^2}\right) e^{-r/3a_0}$ |
| 3 | 1 | 0 | $\frac{1}{\sqrt{2\pi}}$ | $\frac{\sqrt{6}}{2} \cos\theta$ | $\frac{4}{81\sqrt{6}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0}$ | $\frac{\sqrt{2}}{81\sqrt{\pi}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0} \cos\theta$ |
| 3 | 1 | ±1 | $\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$ | $\frac{\sqrt{3}}{2} \sin\theta$ | $\frac{4}{81\sqrt{6}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0}$ | $\frac{1}{81\sqrt{\pi}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0} \sin\theta\, e^{\pm i\phi}$ |
| 3 | 2 | 0 | $\frac{1}{\sqrt{2\pi}}$ | $\frac{\sqrt{10}}{4} (3\cos^2\theta - 1)$ | $\frac{4}{81\sqrt{30}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0}$ | $\frac{1}{81\sqrt{6\pi}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0} (3\cos^2\theta - 1)$ |
| 3 | 2 | ±1 | $\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$ | $\frac{\sqrt{15}}{2} \sin\theta\cos\theta$ | $\frac{4}{81\sqrt{30}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0}$ | $\frac{1}{81\sqrt{\pi}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0} \sin\theta\cos\theta\, e^{\pm i\phi}$ |
| 3 | 2 | ±2 | $\frac{1}{\sqrt{2\pi}} e^{\pm 2i\phi}$ | $\frac{\sqrt{15}}{4} \sin^2\theta$ | $\frac{4}{81\sqrt{30}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0}$ | $\frac{1}{162\sqrt{\pi}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0} \sin^2\theta\, e^{\pm 2i\phi}$ |

*The quantity $a_0 = 4\pi\epsilon_0\hbar^2/me^2 = 5.292 \times 10^{-11}$ m is equal to the radius of the innermost Bohr orbit.

Figure 2: Normalised wave functions of the hydrogen atom for $n = 1, 2$ and 3.