University of the Witwatersrand
Physics IIE (Engineering) : PHYS284 : 2003
Examination : June 2003

Instructions: Answer all questions.

Time: 2 hours = 120 minutes

Total Marks: 120 marks

1. a) Given that the Lorentz Transformation is :

$$\begin{aligned} x' &= \frac{(x-vt)}{\sqrt{1-v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{(t-\frac{vx}{c^2})}{\sqrt{1-v^2/c^2}} \end{aligned} \quad \text{and} \quad \begin{aligned} x &= \frac{(x'+vt')}{\sqrt{1-v^2/c^2}} \\ y &= y' \\ z &= z' \\ t &= \frac{(t'+\frac{vx'}{c^2})}{\sqrt{1-v^2/c^2}} \end{aligned}$$

derive the formula for the relativistic time dilation

$$t = t_0/\sqrt{1-v^2/c^2}.$$

(6)

b) The Global Positioning System (GPS) consists of a network of 24 satellites in roughly 12-hour orbits, each carrying atomic clocks on board. The orbital radius of the satellites is about 26,600 km. The satellites have orbital speeds of about 3.9 km/s in a frame centred on the Earth. The on-board atomic clocks have a period of about 1 nanosecond (ns) and they have a rate accuracy of 1 ns/day. The GPS receiver determines its current position and heading by comparing the time signals it receives via a radio transmission from a number of the GPS satellites (usually 4 to 12) and triangulating on the known positions of each satellite.

i) Consider the distance that light can travel in 1 ns and determine the accuracy with which the timing information from the radio transmisions must be known in order to correspond to the typical position accuracy of 5m. (3)

The timing signals of the clocks are affected by both General and Special Relativity. (The effects of General Relativity are about nine times larger than the those of Special Relativity, but they are not considered here.)

ii) Considering the Special Relativity effects, will the satellite based clock tick slower or faster than the ground based clock as seen from the ground based clock ? Elaborate on your answer briefly. (3)

iii) Now show that the time difference between the satellite based clock, and the ground based clock, amounts to about 7 μs per day. (4)

iv) What positional error does this time error correspond to ? (3)

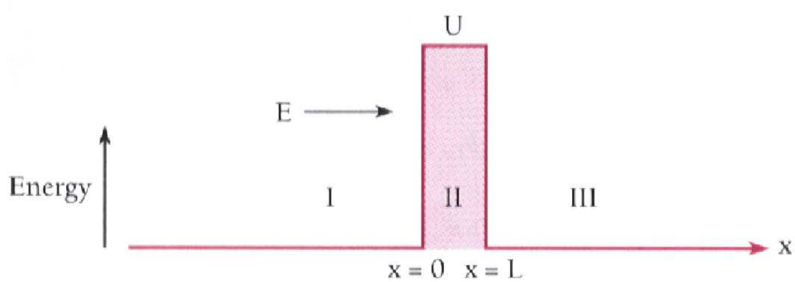
Engineers preset the satellite clocks to new tick rates which account for the effects of both General and Special Relativity. This is done in such a way that once the satellites are in their known orbits with their known speeds, they then remain in better synchronisation with the earth bound clocks. A regular signal corrects any remaining error in their mutual synchronisation. A further relativistic calculation is required by the GPS receiver, once it has acquired four or more satellites, to make additional clock corrections.

v) Consider what this calculation may take into account. (3)

vi) Discuss the sources of error in a GPS system. (3)

Total for Question 1 [25]

2. A quantum particle approaches a rectangular barrier whose height is greater than the energy of the particle as in the figure below.



Consider the Schrödinger Wave Equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + U(x)\psi(x) = E\psi(x)$$

with the potential

$$U(x) = \begin{cases} 0 & x < 0 & \text{region I} \\ U & 0 \leq x \leq L & \text{region II} \\ 0 & x > L & \text{region III} \end{cases} .$$

The solutions will have an oscillatory or an exponential form

$$\psi_\alpha = A_\alpha e^{ik_\alpha x} + B_\alpha e^{-ik_\alpha x} \qquad \text{and} \qquad \psi_\alpha = A_\alpha e^{k_\alpha x} + B_\alpha e^{-k_\alpha x}$$

where $\alpha = I, II$ or III depending on the particular region of the figure being considered.

a) Specify which of the forms belong to which regions and give the structure of the k_α terms. (6)

b) Applying matching conditions at the region interfaces as well as boundary conditions, we find the transmission probability through the barrier is

$$T = e^{-2k_{II}L}$$

Deduce the dependence of the transmission on particle energy, barrier height and barrier width. What is the name given to this purely quantum process ? (4)

c) Discuss the flow of electrons between two partly oxidised contacts, as they approach each other, as in a simple switch, in terms of this theory. (4)

(d) The Scanning Tunneling Microscope directly exploits these insights.

i) Explain the principle of operation by referring to a particle energy drawing showing clearly the three regions corresponding to the sample, the tip and the air-gap in an idealised barrier penetration problem. (4)

ii) The transmission probability of electrons through the barrier in a tunneling microscope is characterised by $k_{II} = 10.0\text{nm}^{-1}$. Suppose the electronic detection system is sensitive to electrical current changes of 0.1%. What surface height sensitivity does this correspond to ? (5)

Total for Question 2 [23]

3. (a) Consider the radial wave equation for the Hydrogen atom,

$$\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left[\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} - \frac{l(l+1)\hbar^2}{2mr^2} - E\right]R = 0$$

as well as the radial wave function for the 1s state

$$R_{1s}(r) = \frac{2}{a_0^{3/2}}e^{-r/a_0}.$$

Show that the Bohr radius and ground state energy for the hydrogen atom are given by

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2} \qquad \text{and} \qquad E_{1s} = -\frac{me^4}{32\pi^2\epsilon_0^2\hbar^2}.$$

(8)

(b) Verify that these two quantities are indeed well predicted, by calculating their values in units of Å and eV respectively. (5)

(c) Verify that the average value of r for a 1s electron is $1.5a_0$ by calculating the expectation value of r given by the equation

$$\langle r \rangle = \int_{allspace} r|\psi|^2 dV$$

with $\psi_{1s}(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) = R_{1s}(r)\frac{1}{2\sqrt{\pi}}$ and with $R_{1s}(r)$ given as above. Also note that

$$dV = r^2 \sin\theta dr\, d\theta\, d\phi \quad \text{and} \quad \Gamma(n+1) = \int_0^\infty \omega^n e^{-\omega} dw = n!$$

(7)

Total for Question 3 [20]

4. (a) Starting with the wave-functions for two identical non-interacting quantum particles, show that the probability for two fermions to occupy the same quantum state is zero, and that the probability for two bosons to occupy the same quantum state is twice that for two classical particles to occupy the same state. (Note : Use $\Psi_C(\mathbf{r}_1,\mathbf{r}_2) = \psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2)$ and $\Psi_{\frac{B}{F}}(\mathbf{r}_1,\mathbf{r}_2) = \frac{1}{\sqrt{2}}[\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) \pm \psi_a(\mathbf{r}_2)\psi_b(\mathbf{r}_1)]$.) (6)

(b) Use this result to state and explain the relative pressure exerted by similar gases of classical molecules, bosons, or fermions for the same temperature. (4)

(c) Sketch the Fermi-Dirac distribution

$$f_{FD}(\epsilon) = \frac{1}{e^{(\epsilon-\epsilon_F)/kT}+1}$$

at low but finite temperature and explain why only electrons near the Fermi-level are expected to participate in transport properties. (4)

(d) Show that if the average occupancy of a state of energy $\epsilon_F + \Delta\epsilon$ is f_1 at any temperature, then the average occupancy of a state of energy $\epsilon_F - \Delta\epsilon$ is $1 - f_1$. (6)

Total for Question 4 [20]

5. The Fermi theory predicts Ohm's law for metallic conductors with the resistivity given as

$$\rho = \left(\frac{mv_F}{ne^2\lambda}\right).$$

Find the mean free path of electrons between collisions in copper at 20°C if its resistivity at that temperature is $\rho = 1.72 \times 10^{-8}\Omega$ m ($\epsilon_F(Cu) = 7.04$eV, $n = 8.46 \times 10^{22}/\text{cm}^3$). (5)

Total for Question 5 [5]

6. (a) Classify metals, semi-conductors and insulators on the basis of their band structure. (6)

b) Consider Si, a semiconductor with a band-gap of 1.11 eV. Sketch the band structure diagram and indicate the position of the Fermi-level. (3)

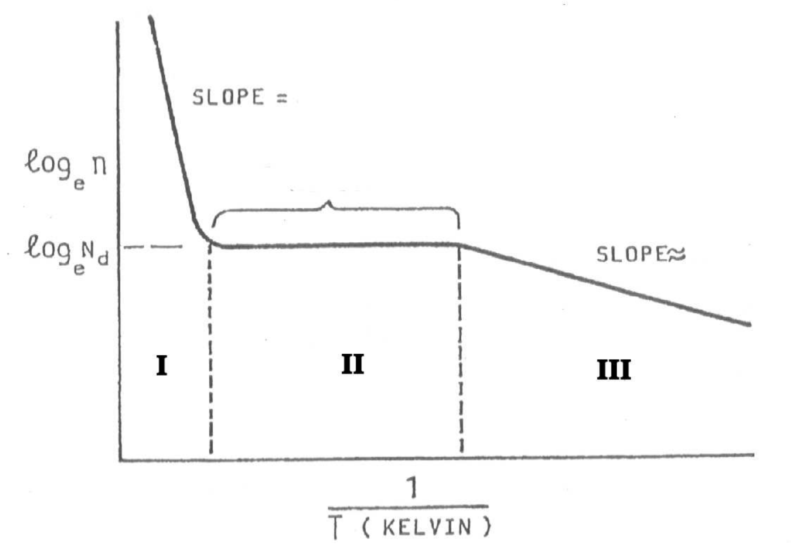
c) The Si is now doped with phosphorous atoms. Using the following data,

$$\epsilon_r = 11.7 \text{ for Si}$$
$$m_e^*/m_e = 0.19$$

calculate the ionisation energy of the phosphorous atom in silicon. (5)

d) Make a new band structure diagram which includes the dopant level and the new position of the Fermi-level. Label fully. (5)

e) The figure below is a logarithmic plot of the charge carrier concentration versus inverse temperature. There are three regions in the plot. Discuss the physics which accounts



for the behaviour of the carrier concentration in each region. Indicate the energy represented by the slope of the straight lines in the regions I and II. (8)

Total for Question 6 [27]

Total Marks **[120]**

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