University of the Witwatersrand
Physics IIE (Engineering) : PHYS284
Examination : June 2005

Instructions: Answer all questions.
Time: 2 hours = 120 minutes
Total Marks: 120 marks (105 marks = 100%)

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1. (a) Given that the Lorentz Transformation is :

$$\begin{aligned} x' &= \frac{(x - vt)}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{(t - \frac{vx}{c^2})}{\sqrt{1 - v^2/c^2}} \end{aligned}$$

derive the formula for the relativistic time dilation

$$t = t_0/\sqrt{1 - v^2/c^2}.$$

(6)

(b) A spacecraft is moving relative to the earth. An observer on earth finds that, according to her clock, 3601s elapse between 1pm and 2pm on the spacecraft's clock. What is the spacecraft's speed relative to the earth ? (6)

(c) Show that the relativistic velocity transformation is

$$v'_x = \frac{v_x - v}{1 - \frac{v_x v}{c^2}}.$$

(5)

(d) Two spaceships, A and B, are approaching each other from opposite directions. An observer on earth measures their velocities to be 0.750c and -0.850c respectively. What is their actual relative velocity, as would be measured by an observer in either ship. (5)

[22]

2. For a particle trapped in a one dimensional box of width L :

(a) Find the average value of the position, $\langle x \rangle$.
(You may need : $2 \sin^2 t = 1 - \cos 2t$.) (6)

(b) Show that the average value of the momentum, $\langle p\rangle = \int \psi^* \mathbf{p} \psi \, dx$ is zero, noting that the momentum operator is $\mathbf{p} = \frac{\hbar}{i}\frac{d}{dx}$.
(You may need : $\int \sin ax \cos ax \, dx = \frac{1}{2a}\sin^2 ax$.) (4)

(c) Explain why this is a logical result. (2)

(d) Treat a quantum dot of dimension 30 nm as an electron in a one dimensional box. The dot is manufactured by using appropriate doping and deposition of contacts on a slab of GaAs and then applying a bias voltage in such a way as to create a linear confining potential. Find the minimum energy of the electron. Work in units of eV for energy and use $\hbar c = 197$ eV.nm.
(Note : for GaAs, $m^* = 0.067 m_e$). (4)

(e) What is the consequence of this result if you require to operate the device in its ground state ? (2)

[18]

3. (a) The probability of finding an electron, of a given state of the hydrogen atom, at some distance between r and $r + dr$ is given by

$$P(r)dr = r^2 |R_{n,l}(r)|^2 \, dr.$$

How much more likely is a $2p$ electron to be at a distance of $4a_0$ from the nucleus than a distance of a_0 ? (Use the attached table in figure 1 below of the normalised wave functions for the hydrogen atom.) (8)

(b) Unsöld's theorem states that for any value of the orbital quantum number l, the probability densities summed over all possible values of m_l are angle independent.

i. Verify this for $l = 1$ by showing $\Sigma_{m_l=-l}^{+l} |\Theta_{l,m_l}|^2 |\Phi_{m_l}|^2 = const$ using the attached table in figure 1 below of the normalised wave functions for the hydrogen atom. (6)

ii. What is the physical meaning of this mathematical fact. (2)

iii. Which group of the periodic table contains elements demonstrating Unsöld's theorem. (2)

iv. How does the mathematical form of the probability density illuminate the properties of these elements ? (2)

[20]

4. (a) i. What are the differences between bosons and fermions quantum mechanically. Give one example of each. (4)

ii. Describe the behaviour of a system of identical bosons and fermions respectively at very low temperatures. (4)

(b) The density of metallic zinc is 7.13 g/cm^3 and the atomic mass of the zinc atom is 65.4 u. The Fermi energy in zinc metal is 11.0 eV. Work out the effective mass of a delocalised electron in Zinc metal. Express your answer in terms of the free electron mass and say why there is a difference.
(Zinc has 2 valence electrons.) (6)

(c) Show for the Fermi-Dirac distribution that if the average occupancy of a state of energy $\epsilon_F + \Delta\epsilon$ is f_1 at any temperature, then the average occupancy of a state of energy $\epsilon_F - \Delta\epsilon$ is $1 - f_1$.
(6)

(d) What is the implication of this for charge carriers in intrinsic material ? (2)

[22]

5. (a) A semiconductor is characterised by the energy band structure. Draw appropriate structures for :

   i. An insulator,
   ii. An intrinsic semiconductor,
   iii. A p-type doped semiconductor,
   iv. A metal.

(4)

(b) Use the expression for the group velocity associated with an electron in a solid as well as Newtons third Law to derive a relationship for the effective mass of the electron in terms of energy and wavenumber. (4)

(c) If the energy dependence of an electron in the valance band is given by $E = E_1(1 - \frac{1}{2}\cos ka)$ where E_1 is a positive constant and a is the lattice spacing,

   i. Sketch the graphs of electron energy, E, the electron velocity v and the effective mass m^* each as a function of k for the valence band from the region $-\frac{\pi}{a} \leq k \leq \frac{\pi}{a}$, (4)
   ii. Determine the hole effective mass at $k = \frac{3\pi}{4a}$ in terms of $\hbar$ and E_1. (4)

(d) What would be the effect of effective mass of the charge carrier on

   i. The appearance of the tunneling limit to miniaturisation.
   ii. The level of the state in the gap.
   iii. The frequency response of the device.

Back up each answer by specifying a formula which demonstrates the physics. (6)

[22]

6. (a) Show that the transmission of a Gaussian laser beam ($f(r) = f_0 e^{-2\frac{r}{w}^2}$) of $1/e^2$ radius w, through a circular aperture of radius a, can be written as:

$$T = T_0 \left(1 - \exp\left(-\frac{2a^2}{w^2}\right)\right)$$

where T is the fraction of energy transmitted through the aperture, and T_0 is the maximum transmission. (6)

(b) The table of data given below was collected for a CO_2 laser beam passing through a circular aperture. Use the data to estimate the $1/e^2$ beam radius w assuming that the laser beam has a Gaussian fluence profile.

Aperture diameter (mm)	**Transmitted energy (mJ)**
1.5	71
3	244
4.0	368
5.0	428
6.0	468
∞	488

(3)

(c) Lasers usually emit in a range of closely spaced wavelengths satisfying the standing wave condition: $m\lambda/2 = L$, where m is an integer and L is the cavity length. Show that the smallest possible interval between wavelengths can be approximated as:

$$\Delta\lambda \approx \lambda^2/2L.$$

(2)

(d) An ultra-short pulsed laser outputs 10nJ pulses of FWHM duration 50fs at 780nm wavelength. If the resonator cavity length is 1m:

i. How many longitudinal modes would have to be oscillating?
ii. What would the gain bandwidth of this laser have to be to support these modes?
iii. At what repetition rate would you expect this laser to operate at?
iv. What would the peak power and average power of the laser be?

(5)

[16]

Total Marks **[120]**

n	l	m_l	$\Phi(\phi)$	$\Theta(\theta)$	$R(r)$	$\psi(r,\theta,\phi)$
1	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{a_0^{3/2}} e^{-r/a_0}$	$\frac{1}{\sqrt{\pi}\, a_0^{3/2}} e^{-r/a_0}$
2	0	0	$\frac{1}{\sqrt{2\pi}}$*	$\frac{1}{\sqrt{2}}$	$\frac{1}{2\sqrt{2}\, a_0^{3/2}}\left(2-\frac{r}{a_0}\right)e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\, a_0^{3/2}}\left(2-\frac{r}{a_0}\right)e^{-r/2a_0}$
2	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2}\cos\theta$	$\frac{1}{2\sqrt{6}\, a_0^{3/2}}\frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\, a_0^{3/2}}\frac{r}{a_0} e^{-r/2a_0}\cos\theta$
2	1	±1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{3}}{2}\sin\theta$	$\frac{1}{2\sqrt{6}\, a_0^{3/2}}\frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{8\sqrt{\pi}\, a_0^{3/2}}\frac{r}{a_0} e^{-r/2a_0}\sin\theta\, e^{\pm i\phi}$
3	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{81\sqrt{3}\, a_0^{3/2}}\left(27-18\frac{r}{a_0}+2\frac{r^2}{a_0^2}\right)e^{-r/3a_0}$	$\frac{1}{81\sqrt{3\pi}\, a_0^{3/2}}\left(27-18\frac{r}{a_0}+2\frac{r^2}{a_0^2}\right)e^{-r/3a_0}$
3	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2}\cos\theta$	$\frac{4}{81\sqrt{6}\, a_0^{3/2}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0} e^{-r/3a_0}$	$\frac{\sqrt{2}}{81\sqrt{\pi}\, a_0^{3/2}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0} e^{-r/3a_0}\cos\theta$
3	1	±1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{3}}{2}\sin\theta$	$\frac{4}{81\sqrt{6}\, a_0^{3/2}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0} e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\, a_0^{3/2}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0} e^{-r/3a_0}\sin\theta\, e^{\pm i\phi}$
3	2	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{10}}{4}(3\cos^2\theta-1)$	$\frac{4}{81\sqrt{30}\, a_0^{3/2}}\frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{1}{81\sqrt{6\pi}\, a_0^{3/2}}\frac{r^2}{a_0^2} e^{-r/3a_0}(3\cos^2\theta-1)$
3	2	±1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{15}}{2}\sin\theta\cos\theta$	$\frac{4}{81\sqrt{30}\, a_0^{3/2}}\frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\, a_0^{3/2}}\frac{r^2}{a_0^2} e^{-r/3a_0}\sin\theta\cos\theta\, e^{\pm i\phi}$
3	2	±2	$\frac{1}{\sqrt{2\pi}} e^{\pm 2i\phi}$	$\frac{\sqrt{15}}{4}\sin^2\theta$	$\frac{4}{81\sqrt{30}\, a_0^{3/2}}\frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{1}{162\sqrt{\pi}\, a_0^{3/2}}\frac{r^2}{a_0^2} e^{-r/3a_0}\sin^2\theta\, e^{\pm 2i\phi}$

*The quantity $a_0 = 4\pi\epsilon_0\hbar^2/me^2 = 5.292 \times 10^{-11}$ m is equal to the radius of the innermost Bohr orbit.

Figure 1: Normalised wave functions of the hydrogen atom for $n = 1, 2$ and 3.

No more pages