University of the Witwatersrand
Physics IIE (Engineering) : PHYS284
Examination : June 2006

Instructions: Answer all questions.
Time: 2 hours = 120 minutes
Total Marks: 120 marks (105 marks = 100%)

---

1. (a) One of the most dramatic early confirmations of the phenomena of length contraction and time dilation is the profusion of muons reaching the surface of the earth. These muons are produced in cosmic ray collisions with the upper atmosphere at altitudes of 6 km and greater. A muon is a lepton, like an electron, but it has a larger mass ($m_\mu \approx 207 m_e$) and it is not stable ($\tau_\mu \approx 2.2\,\mu$s). The muons are produced in the upper atmosphere with speeds $v_\mu \approx 0.995c$.

   i. Show that a non-relativistic analysis leads to the conclusion that very few muons would survive long enough to reach the surface of the earth (sea level). (3)

   ii. Calculate the relativistic gamma parameter. (3)

   iii. In actual fact, experiment shows most of the muons reach the surface of the earth. Explain how this occurs considering the viewpoint of an observer on earth. (5)

   iv. Show that a hypothetical observer travelling with the muon would agree with the earth bound observer. (5)

   (Hint : You will need to use time dilation in the one case, and length contraction in the other, but you must justify your choice, as well as perform the calculation.)

   (b) i. From the relativistic expressions for total energy and momentum, derive the result

$$E^2 - p^2c^2 = m_0^2c^4.$$

(5)

   ii. Verify that

$$v/c = pc/E.$$

(4)

Total for Question 1 [25]

2. (a) With regard to the Scanning Tunneling Microscope :

i. Explain the principle of operation by referring to a particle energy drawing showing clearly the three regions corresponding to the sample, the tip and the air-gap in an idealised barrier penetration problem. (4)

ii. The transmission coefficient of quantum particles through the barrier is given by

$$T = e^{-2\frac{\sqrt{2m(U-E)}}{\hbar}L}.$$

Indicate the meaning of the symbols $(U - E)$, L and m. (3)

iii. For a specific tunneling microscope, we can write

$$T = e^{-2CL}$$

where $C = 10.0\text{nm}^{-1}$. Suppose the electronic detection system is sensitive to electrical current changes of 0.1%. What surface height sensitivity does this correspond to. (4)

(b) The tunneling process also represents a physics limit for the miniaturisation of features on a chip. This could be either tunneling between neighbouring wires or across the gate of a transistor. The corresponding classical process is thermally stimulated over-barrier hopping.

i. Imagine that the quantum process should not be more likely than the classical process, so that the limiting case is when they are equal. This will allow you to estimate a value for the minimum feature size L of a conventional chip. Estimate realistic values for $(U - E)$ and kT and explain your choice. (4)

ii. Perform the calculation to find L. (5)

Total for Question 2 [20]

3. (a) Consider the radial wave equation for the Hydrogen atom,

$$\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left[\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}-\frac{l(l+1)\hbar^2}{2mr^2}-E\right]R=0$$

as well as the radial wave function for the 1s state (see the table at the end of the paper).

i. Show that the Bohr radius and ground state energy for the hydrogen atom are given by

$$a_0=\frac{4\pi\epsilon_0\hbar^2}{me^2} \qquad \text{and} \qquad E_{1s}=-\frac{me^4}{32\pi^2\epsilon_0^2\hbar^2}.$$

(6)

ii. Verify that these two quantities are indeed well predicted, by calculating their values in units of Å and eV respectively. (4)

(b) The energy levels of certain kinds of two-electron atoms/ions of atomic number Z may be approximated by

$$E_n(Z)=E_1(H)Z^2-\frac{E_1(H)(Z-1)^2}{n^2}$$

where the ground state of hydrogen is $E_1(H)=-13.6\text{eV}$.

i. Sketch the two-electron atom/ion under conditions where the above expression could be expected to hold. (4)

ii. Using the full expression for quantised energy levels in hydrogen-like (that is, one-electron) atoms/ions of atomic number Z, describe how each term in the above expression arises. (4)

iii. Why does the accuracy increase when n increases ? (2)

Total for Question 3 [20]

4. (a) Show that the median energy in a free electron gas at $T = 0$ is $\epsilon_F/2^{2/3} = 0.630\epsilon_F$. (Hint: The median energy defines the point where the total number of states below this value is equal to the total number of states above this value.) (4)

(b) The Fermi energy in copper is 7.04 eV. Compare the approximate average energy of electrons in copper at room temperature ($kT = 0.025$ eV) with what their average energy would have been as classical particles. (4)

(c) Only electrons near the Fermi energy may participate in transport. If the mean free path of conduction electrons is given by λ, the average time between collisions is $\tau = \lambda/v_F$.

i. Use this result to show that the drift velocity of conduction electrons will be

$$v_d = \frac{V}{L}\frac{e}{m} \cdot \frac{\lambda}{v_F}.$$

(4)

ii. A microscopic model for current in a conductor gives $I = nAev_d$. Derive Ohm's Law and show that the resistivity may be given by

$$\rho = \left(\frac{mv_F}{ne^2\lambda}\right).$$

(4)

iii. The resistivity of copper is $\rho = 1.72 \times 10^{-8}\Omega$ m and the Fermi velocity is $v_F =$ 0.52%c. Show that the mean free path between collisions for an electron in copper is 38.3 nm.
(For copper, $\mathcal{M} = 63.5$g/mol and $\rho = 8.96$g/cm^{-3}.) (4)

Total for Question 4 [20]

5/Page five...

5. (a) In the case of free electrons in a three dimensional box (a Fermi gas) we find the number of energy states per energy interval is:

$$n(\epsilon)d\epsilon = \frac{8\sqrt{2}\pi L^3 m^{3/2}}{h^3} \frac{\sqrt{\epsilon}d\epsilon}{e^{(\epsilon-\epsilon_F)/kT}+1}.$$

Make three changes to this expression to adjust it for the context of a semiconductor, and the derive the equation for number of charge carriers in the conduction band in an intrinsic semi-conductor.

$$n_i = 2V\left(\frac{m^* kT}{2\pi\hbar^2}\right)^{3/2} e^{(\epsilon_i-\epsilon_g)/kT},$$

where $\epsilon_i \approx \epsilon_g/2$ is the Fermi level in an intrinsic semiconductor.
(Hint: $\int_0^\infty x^{1/2}e^{-x}\,dx = \Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$). (4)

(b) A slab of silicon is doped with the phosphorous atoms. Treating the last electron as hydrogenic around an effective Si^{1+} core, calculate the binding energy of this electron. (Use $m^* = 0.19m$ and $\epsilon_r = 11.7$). (4)

(c) Sketch the band structure of Silicon, include a labelled energy axis, indicate the position of the P donor level on this diagram. .
(Use $\epsilon_g = 1.1$). (4)

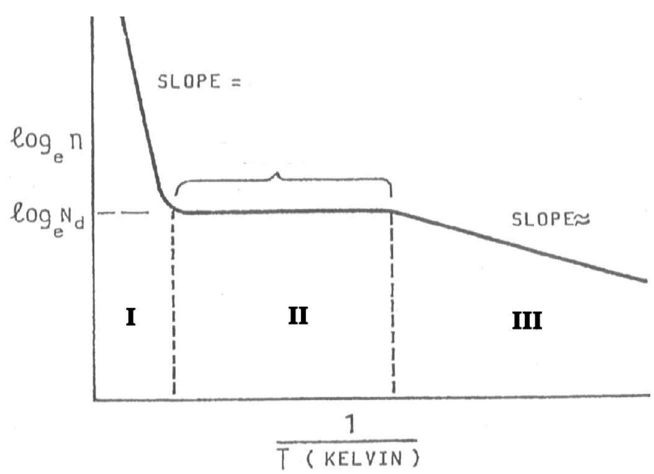
(d) Discuss the logic behind your choice of where to place the P donor level. (2)

(e) Using similar arguments and mathematics as in the intrinsic case, we find the Fermi level in the case of n-type material :

$$n_n = n_i \exp\left(\frac{\epsilon_F-\epsilon_i}{kT}\right) \qquad \text{with} \qquad \epsilon_F \approx \frac{\epsilon_d+\epsilon_g}{2}$$

Hence explain in detail the regions I, II and III in the graph below (inluding a calculation of the slopes). (6)

Total for Question 5 [20]



6. The propagation of a laser beam after a lens is given by the following:

$$\omega^2(z) = \left(1 - \frac{z}{f}\right)^2 \omega_0^2 + \left(z + \left(1 - \frac{z}{f}\right) d\right)^2 \theta_0^2$$

where ω_0 and θ_0 are the beam waist and divergence prior to the lens (of focal length f), d is the distance from ω_0 to the lens, and z is the distance from the lens.

(a) Show that the waist position after the lens is given by

$$z = \frac{f(\omega_0^2 + \theta_0^2(d - f)d)}{\omega_0^2 + \theta_0^2(d - f)^2}$$

(Hint, note that $\frac{d\omega^2}{dz} = 2\omega\frac{d\omega}{dz}$) (6)

(b) Show under what conditions the geometric approximations will be true (i.e., that the new waist position is at the focal plane of the lens). (3)

(c) If a 193nm laser with a Gaussian beam waist of $\omega_0 = 1\,\text{mm}$ is used for photolithography by passing it through a lens of focal length 0.1m, what will the smallest area that can be irradiated be at the focal plane of the lens (assume a circular beam)? What would you change in the optical system (lens and laser) to decrease the feature sizes etched by the laser beam? (6)

Total for Question 6 [15]

Total Marks **[120]**

n	l	m_l	$\Phi(\phi)$	$\Theta(\theta)$	$R(r)$	$\psi(r, \theta, \phi)$
1	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{a_0^{3/2}} e^{-r/a_0}$	$\frac{1}{\sqrt{\pi}\, a_0^{3/2}} e^{-r/a_0}$
2	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2\sqrt{2}\, a_0^{3/2}} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\, a_0^{3/2}} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$
2	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2} \cos\theta$	$\frac{1}{2\sqrt{6}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \cos\theta$
2	1	±1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{3}}{2} \sin\theta$	$\frac{1}{2\sqrt{6}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{8\sqrt{\pi}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \sin\theta\, e^{\pm i\phi}$
3	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{81\sqrt{3}\, a_0^{3/2}} \left(27 - 18\frac{r}{a_0} + 2\frac{r^2}{a_0^2}\right) e^{-r/3a_0}$	$\frac{1}{81\sqrt{3\pi}\, a_0^{3/2}} \left(27 - 18\frac{r}{a_0} + 2\frac{r^2}{a_0^2}\right) e^{-r/3a_0}$
3	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2} \cos\theta$	$\frac{4}{81\sqrt{6}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0}$	$\frac{\sqrt{2}}{81\sqrt{\pi}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0} \cos\theta$
3	1	±1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{3}}{2} \sin\theta$	$\frac{4}{81\sqrt{6}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0} \sin\theta\, e^{\pm i\phi}$
3	2	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{10}}{4} (3\cos^2\theta - 1)$	$\frac{4}{81\sqrt{30}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{1}{81\sqrt{6\pi}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0} (3\cos^2\theta - 1)$
3	2	±1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{15}}{2} \sin\theta\cos\theta$	$\frac{4}{81\sqrt{30}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0} \sin\theta\cos\theta\, e^{\pm i\phi}$
3	2	±2	$\frac{1}{\sqrt{2\pi}} e^{\pm 2i\phi}$	$\frac{\sqrt{15}}{4} \sin^2\theta$	$\frac{4}{81\sqrt{30}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{1}{162\sqrt{\pi}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0} \sin^2\theta\, e^{\pm 2i\phi}$

*The quantity $a_0 = 4\pi\epsilon_0\hbar^2/me^2 = 5.292 \times 10^{-11}$ m is equal to the radius of the innermost Bohr orbit.

Figure 1: Normalised wave functions of the hydrogen atom for $n = 1, 2$ and 3.

No more pages